80.1 Introduction

Signals from sensors do not usually have suitable characteristics for display, recording, transmission, or further processing. For example, they may lack the amplitude, power, level, or bandwidth required, or they may carry superimposed interference that masks the desired information.

Signal conditioners, including amplifiers, adapt sensor signals to the requirements of the receiver (circuit or equipment) to which they are to be connected. The functions to be performed by the signal conditioner derive from the nature of both the signal and the receiver. Commonly, the receiver requires a single-ended, low-frequency (dc) voltage with low output impedance and amplitude range close to its power-supply voltage(s). A typical receiver here is an analog-to-digital converter (ADC).

Signals from sensors can be analog or digital. Digital signals come from position encoders, switches, or oscillator-based sensors connected to frequency counters. The amplitude for digital signals must be compatible with logic levels for the digital receiver, and their edges must be fast enough to prevent any false triggering. Large voltages can be attenuated by a voltage divider and slow edges can be accelerated by a Schmitt trigger.

Analog sensors are either self-generating or modulating. Self-generating sensors yield a voltage (thermocouples, photovoltaic, and electrochemical sensors) or current (piezo- and pyroelectric sensors) whose
bandwidth equals that of the measurand. Modulating sensors yield a variation in resistance, capacitance, self-inductance or mutual inductance, or other electrical quantities. Modulating sensors need to be excited or biased (semiconductor junction-based sensors) in order to provide an output voltage or current. Impedance variation-based sensors are normally placed in voltage dividers, or in Wheatstone bridges (resistive sensors) or ac bridges (resistive and reactance-variation sensors). The bandwidth for signals from modulating sensors equals that of the measured in dc-excited or biased sensors, and is twice that of the measurand in ac-excited sensors (sidebands about the carrier frequency) (see Chapter 81). Capacitive and inductive sensors require an ac excitation, whose frequency must be at least ten times higher than the maximal frequency variation of the measurand. Pallás-Areny and Webster [1] give the equivalent circuit for different sensors and analyze their interface.

Current signals can be converted into voltage signals by inserting a series resistor into the circuit. Graeme [2] analyzes current-to-voltage converters for photodiodes, applicable to other sources. Henceforth, we will refer to voltage signals to analyze transformations to be performed by signal conditioners.

80.2 Dynamic Range

The dynamic range for a measurand is the quotient between the measurement range and the desired resolution. Any stage for processing the signal form a sensor must have a dynamic range equal to or larger than that of the measurand. For example, to measure a temperature from 0 to 100°C with 0.1°C resolution, we need a dynamic range of at least \( (100 - 0)/0.1 = 1000 \) (60 dB). Hence a 10-bit ADC should be appropriate to digitize the signal because \( 2^{10} = 1024 \). Let us assume we have a 10-bit ADC whose input range is 0 to 10 V; its resolution will be 10 V/1024 = 9.8 mV. If the sensor sensitivity is 10 mV/°C and we connect it to the ADC, the 9.8 mV resolution for the ADC will result in a 9.8 mV/(10 mV/°C) = 0.98°C resolution! In spite of having the suitable dynamic range, we do not achieve the desired resolution in temperature because the output range of our sensor (0 to 1 V) does not match the input range for the ADC (0 to 10 V).

The basic function of voltage amplifiers is to amplify the input signal so that its output extends across the input range of the subsequent stage. In the above example, an amplifier with a gain of 10 would match the sensor output range to the ADC input range. In addition, the output of the amplifier should depend only on the input signal, and the signal source should not be disturbed when connecting the amplifier. These requirements can be fulfilled by choosing the appropriate amplifier depending on the characteristics of the input signal.

80.3 Signal Classification

Signals can be classified according to their amplitude level, the relationship between their source terminals and ground, their bandwidth, and the value of their output impedance. Signals lower than around 100 mV are considered to be low level and need amplification. Larger signals may also need amplification depending on the input range of the receiver.

Single-Ended and Differential Signals

A single-ended signal source has one of its two output terminals at a constant voltage. For example, Figure 80.1a shows a voltage divider whose terminal L remains at the power-supply reference voltage regardless of the sensor resistance, as shown in Figure 80.1b. If terminal L is at ground potential (grounded power supply in Figure 80.1a), then the signal is single ended and grounded. If terminal L is isolated from ground (for example, if the power supply is a battery), then the signal is single ended and floating. If terminal L is at a constant voltage with respect to ground, then the signal is single ended and driven off ground. The voltage at terminal H will be the sum of the signal plus the off-ground voltage. Therefore, the off-ground voltage is common to H and L; hence, it is called the common-mode voltage. For example, a thermocouple bonded to a power transistor provides a signal whose amplitude depends on the temperature of the transistor case, riding on a common-mode voltage equal to the case voltage.
A differential signal source has two output terminals whose voltages change simultaneously by the same magnitude but in opposite directions. The Wheatstone bridge in Figure 80.1c provides a differential signal which is the difference between two voltages \( v_H \) and \( v_L \) having the same amplitude but opposite signs and riding on a common-mode voltage \( V_c \). For differential signals much smaller than the common-mode voltage, the equivalent circuit in (e) is used. If the reference point is grounded, the signal (single-ended or differential) will be grounded; if the reference point is floating, the signal will also be floating.

A differential signal source has two output terminals whose voltages change simultaneously by the same magnitude but in opposite directions. The Wheatstone bridge in Figure 80.1c provides a differential signal. Its equivalent circuit (Figure 80.1d) shows that there is a differential voltage \( (v_d = v_H - v_L) \) proportional to \( x \) and a common-mode voltage \( (V_c = V/2) \) that does not carry any information about \( x \). Further, the two output impedances are balanced. We thus have a balanced differential signal with a superimposed common-mode voltage. Were the output impedances different, the signal would be unbalanced. If the bridge power supply is grounded, then the differential signal will be grounded; otherwise, it will be floating. When the differential signal is very small as compared with the common-mode voltage, in order to simplify circuit analysis it is common to use the equivalent circuit in Figure 80.1e. Some differential signals (grounded or floating) do not bear any common-mode voltage.
Signal conditioning must ensure the compatibility between sensor signals and receivers, which will depend on the relationship between input terminals and ground. For example, a differential and grounded signal is incompatible with an amplifier having a grounded input terminal. Hence, amplifiers must also be described according to their input topology.

**Narrowband and Broadband Signals**

A narrowband signal has a very small frequency range relative to its central frequency. Narrowband signals can be dc, or static, resulting in very low frequencies, such as those from a thermocouple or a weighing...
scale, or ac, such as those from an ac-driven modulating sensor, in which case the exciting frequency (carrier) becomes the central frequency (see Chapter 81).

Broadband signals, such as those from sound and vibration sensors, have a large frequency range relative to their central frequency. Therefore, the value of the central frequency is crucial; a signal ranging from 1 Hz to 10 kHz is a broadband instrumentation signal, but two 10 kHz sidebands around 1 MHz are considered to be a narrowband signal. Signal conditioning of ac narrowband signals is easier because the conditioner performance only needs to be guaranteed with regard to the carrier frequency.

**Low- and High-Output-Impedance Signals**

The output impedance of signals determines the requirements of the input impedance of the signal conditioner. Figure 80.2a shows a voltage signal connected to a device whose input impedance is $Z_d$. The voltage detected will be

\[ v_d = v_i \frac{Z_d}{Z_d + Z_s} \]  

(80.1)

Therefore, the voltage detected will equal the signal voltage only when $Z_d \gg Z_s$; otherwise $v_d \neq v_i$ and there will be a *loading effect*. Furthermore, it may happen that a low $Z_d$ disturbs the sensor, changing the value of $v_i$ and rendering the measurement useless or, worse still, damaging the sensor.

At low frequencies, it is relatively easy to achieve large input impedances even for high-output-impedance signals, such as those from piezoelectric sensors. At high frequencies, however, stray input capacitances make it more difficult. For narrowband signals this is not a problem because the value for $Z_s$ and $Z_d$ will be almost constant and any attenuation because of a loading effect can be taken into account later. However, if the impedance seen by broadband signals is frequency dependent, then each frequency signal undergoes different attenuations which are impossible to compensate for.

Signals with very high output impedance are better modeled as current sources, Figure 80.2b. The current through the detector will be
In order for $i_d = i_s$, it is required that $Z_d \ll Z_s$ which is easier to achieve than $Z_d \gg Z_s$. If $Z_d$ is not low enough, then there is a shunting effect.

$$i_d = i_s \frac{Z_s}{Z_d + Z_s}$$

(80.2)

In order for $i_d = i_s$, it is required that $Z_d \ll Z_s$ which is easier to achieve than $Z_d \gg Z_s$. If $Z_d$ is not low enough, then there is a shunting effect.

## 80.4 General Amplifier Parameters

A **voltage amplifier** produces an output voltage which is a proportional reproduction of the voltage difference at its input terminals, regardless of any common-mode voltage and without loading the voltage source. **Figure 80.3a** shows the equivalent circuit for a general (differential) amplifier. If one input terminal is connected to one output terminal as in **Figure 80.3b**, the amplifier is single ended; if this common terminal is grounded, the amplifier is single ended and grounded; if the common terminal is isolated from ground, the amplifier is single ended and floating. In any case, the output power comes from the power supply, and the input signal only controls the shape of the output signal, whose amplitude is determined by the **amplifier gain**, defined as

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FIGURE 80.3  General amplifier, differential (a) or single ended (b). The input voltage controls the amplitude of the output voltage, whose power comes from the power supply.
The ideal amplifier would have any required gain for all signal frequencies. A practical amplifier has a gain that rolls off at high frequency because of parasitic capacitances. In order to reduce noise and reject interference, it is common to add reactive components to reduce the gain for out-of-band frequencies further. If the gain decreases by \( n \) times 10 when the frequency increases by 10, we say that the gain (downward) slope is 20\( n \) dB/decade. The corner (or –3 dB) frequency \( f_0 \) for the amplifier is that for which the gain is 70% of that in the bandpass. (Note: 20 \log 0.7 = –3 dB). The gain error at \( f_0 \) is then 30%, which is too large for many applications. If a maximal error \( \varepsilon \) is accepted at a given frequency \( f \), then the corner frequency for the amplifier should be

\[
f_0 = \frac{f(1-\varepsilon)}{\sqrt{2\varepsilon - \varepsilon^2}} = \frac{f}{\sqrt{2\varepsilon}}
\]

For example, \( \varepsilon = 0.01 \) requires \( f_0 = 7f \); \( \varepsilon = 0.001 \) requires \( f_0 = 22.4f \). A broadband signal with frequency components larger than \( f \) would undergo amplitude distortion. A narrowband signal centered on a frequency larger than \( f \) would be amplified by a gain lower than expected, but if the actual gain is measured, the gain error can later be corrected.

Whenever the gain decreases, the output signal is delayed with respect to the output. In the above amplifier, an input sine wave of frequency \( f_0 \) will result in an output sine wave delayed by 45° (and with relative attenuation 30% as compared with a sine wave of frequency \( f \gg f_0 \)). Complex waveforms having frequency components close to \( f_0 \) would undergo shape (or phase) distortion. In order for a waveform to be faithfully reproduced at the output, the phase delay should be either zero or proportional to the frequency (linear phase shift). This last requirement is difficult to meet. Hence, for broadband signals it is common to design amplifiers whose bandwidth is larger than the maximal input frequency. Narrowband signals undergo a delay which can be measured and corrected.

An ideal amplifier would have infinite input impedance. Then no input current would flow when connecting the signal, Figure 80.2a, and no energy would be taken from the signal source, which would remain undisturbed. A practical amplifier, however, will have a finite, yet large, input impedance at low frequencies, decreasing at larger frequencies because of stray input capacitances. If sensors are connected to conditioners by coaxial cables with grounded shields, then the capacitance to ground can be very large (from 70 to 100 pF/m depending on the cable diameter). This capacitance can be reduced by using driven shields (or guards) (see Chapter 89). If twisted pairs are used instead, the capacitance between wires is only about 5 to 8 pF/m, but there is an increased risk of capacitive interference.

Signal conditioners connected to remote sensors must be protected by limiting both voltage and input currents. Current can be limited by inserting a power resistor (100 \Omega to 1 k\Omega, 1 W for example), a PTC resistor or a fuse between each signal source lead and conditioner input. Input voltages can be limited by connecting diodes, zeners, metal-oxide varistors, gas-discharge devices, or other surge-suppression nonlinear devices, from each input line to dc power-supply lines or to ground, depending on the particular protecting device. Some commercial voltage limiters are Thyzorb® and Transzorb® (General Semiconductor), Transil® and Trisil® (SGS-Thomson), SIOV® (Siemens), and TL7726 (Texas Instruments).

The ideal amplifier would also have zero output impedance. This would imply no loading effect because of a possible finite input impedance for the following stage, low output noise, and unlimited output power. Practical amplifiers can indeed have a low output impedance and low noise, but their output power is very limited. Common signal amplifiers provide at best about 40 mA output current and sometimes only 10 mA. The power gain, however, is quite noticeable, as input currents can be in the picoampere range (10⁻¹² A) and input voltages in the millivolt range (10⁻³ V); a 10 V, 10 mA output would mean a power gain of 10¹⁴! Yet the output power available is very small (100 mW). Power amplifiers
are quite the opposite; they have a relatively small power gain but provide a high-power output. For both signal and power amplifiers, output power comes from the power supply, not from the input signal.

Some sensor signals do not require amplification but only impedance transformation, for example, to match their output impedance to that of a transmission line. Amplifiers for impedance transformation (or matching) and $G = 1$ are called buffers.

### 80.5 Instrumentation Amplifiers

For instrumentation signals, the so-called instrumentation amplifier (IA) offers performance closest to the ideal amplifier, at a moderate cost (from $1.50 up). Figure 80.4a shows the symbol for the IA and Figure 80.4b its input/output relationship; ideally this is a straight line with slope $G$ and passing through the point (0,0), but actually it is an off-zero, seemingly straight line, whose slope is somewhat different from $G$. The output voltage is

$$v_o = v_a + \left( v_a + v_b + v_c \right) G + v_{ref} \tag{80.5}$$

where $v_a$ depends on the input voltage $v_a$, the second term includes offset, drift, noise, and interference-rejection errors, $G$ is the designed gain, and $v_{ref}$ is the reference voltage, commonly 0 V (but not necessarily, thus allowing output level shifting). Equation 80.5 describes a worst-case situation where absolute values for error sources are added. In practice, some cancellation between different error sources may happen.

Figure 80.5 shows a circuit model for error analysis when a practical IA is connected to a signal source (assumed to be differential for completeness). Impedance from each input terminal to ground ($Z_c$) and between input terminals ($Z_d$) are all finite. Furthermore, if the input terminals are both connected to ground, $v_o$ is not zero and depends on $G$; this is modeled by $V_{os}$. If the input terminals are grounded through resistors, then $v_o$ also depends on the value of these resistors; this is modeled by current sources $I_{os}$ and $I_{os}$, which represent input bias or leakage currents. These currents need a return path, and therefore a third lead connecting the signal source to the amplifier, or a common ground, is required. Neither $V_{os}$ nor $I_{os}$ nor $I_{os}$ is constant; rather, they change with temperature and time: slow changes (<0.01 Hz) are called drift and fast changes are described as noise (hence the noise sources $e_v$, $i_{so}$, and $i_{so}$ in Figure 80.5). Common specifications for IAs are defined in Reference 3.

If a voltage $v_c$ is simultaneously applied to both inputs, then $v_o$ depends on $v_c$ and its frequency. The common-mode gain is

$$G_c(f) = \frac{V_o(v_c = 0)}{V_c} \tag{80.6}$$

In order to describe the output voltage due to $v_c$ as an input error voltage, we must divide the corresponding $v_o(v_c)$ by $G$ (the normal- or differential-mode gain, $G = G_d$). The common-mode rejection ratio (CMRR) is defined as

$$\text{CMRR} = \frac{G_d(f)}{G_c(f)} \tag{80.7}$$

and is usually expressed in decibels ($\text{CMRR}_\text{dB} = 20 \log \text{CMRR}$). The input error voltage will be

$$v_o(v_c) = \frac{G_d v_c}{G_d} = \frac{v_c}{\text{CMRR}} \tag{80.8}$$
In the above analysis we have assumed $Z_c \ll R_o$; otherwise, if there were any unbalance (such as that for the source impedance in Figure 80.5), $v_c$ at the voltage source would result in a differential-mode voltage at the amplifier input,

$$v_d(v_c) = v_c \left( \frac{R_o + \Delta R_o}{Z_c + R_o + \Delta R_o} - \frac{R_o}{Z_c + R_o} \right)$$  \hspace{1cm} (80.9)
which would be amplified by $G$. Then, the effective common-mode rejection ratio would be

$$CMRR_e = CMRR + \frac{1}{Z_c} \frac{\Delta R_n}{\text{CMRR}}$$

where the CMRR is that of the IA alone, expressed as a fraction, not in decibels. Stray capacitances from input terminals to ground will decrease $Z_c$, therefore reducing CMRR$_e$.

The ideal amplifier is unaffected by power supply fluctuations. The practical amplifier shows output fluctuations when supply voltages change. For slow changes, the equivalent input error can be expressed as a change in input offset voltages in terms of the power supply rejection ratio (PSRR),

$$PSRR = \frac{\Delta V_{os}}{\Delta V_i}$$

The terms in Equation 80.5 can be detailed as follows. Because of gain errors we have

$$v_i = v_i \left( G + e_G + \frac{\Delta G}{\Delta T} \times \Delta T + e_{NLG} \right)$$

where $G$ is the differential gain designed, $e_G$ its absolute error, $\Delta G/\Delta T$ its thermal drift, $\Delta T$ the difference between the actual temperature and that at which the gain $G$ is specified, and $e_{NLG}$ is the nonlinearity gain error, which describes the extent to which the input/output relationship deviates from a straight

**FIGURE 80.5** A model for a practical instrumentation amplifier including major error sources.
The actual temperature $T_J$ is calculated by adding to the current ambient temperature $T_A$ the temperature rise produced by the power $P_D$ dissipated in the device. This rise depends on the thermal resistance $q_{JA}$ for the case

$$T_J = T_A + P_D \times q_{JA} \tag{80.13}$$

where $P_D$ can be calculated from the respective voltage and current supplies

$$P_D = |V_{DS}|I_{DS} + |V_{DS}|I_{DS} \tag{80.14}$$

The terms for the equivalent input offset error will be

$$v_{oa} = V_{oa}(T_a) + \frac{\Delta V_{oa}}{\Delta T} \times (T_j - T_a) \tag{80.15}$$

$$v_b = (I_{b+} - I_{b-})R_o + I_{b+} \Delta R_o = I_{oa} R_o + I_{b} \Delta R_o \tag{80.16}$$

where $T_a$ is the ambient temperature in data sheets, $I_{oa} = I_{b+} - I_{b-}$ is the offset current, $I_b = (I_{b+} + I_{b-})/2$, and all input currents must be calculated at the actual temperature,

$$I = I(T_a) + \frac{\Delta I}{\Delta T} \times (T_j - T_a) \tag{80.17}$$

Error contributions from finite interference rejection are

$$v_i = \frac{v_c}{\text{CMRR}_c} + \frac{\Delta V_c}{\text{PSRR}} \tag{80.18}$$

where the CMRR must be that at the frequency for $v_c$, and the PSRR must be that for the frequency of the ripple $\Delta V_c$. It is assumed that both frequencies fall inside the bandpass for the signal of interest $v_d$.

The equivalent input voltage noise is

$$v_n = \sqrt{\frac{e_n^2 R_o + i_n^2 - R^2 B_{+} + i_n^2 - R^2 B_{-}}{B_{e}}} \tag{80.19}$$

where $e_n^2$ is the voltage noise power spectral density of the IA, $i_n^2$ and $i_n^2$ are the current noise power spectral densities for each input of the IA, and $B_o$, $B_+$, and $B_-$ are the respective noise equivalent bandwidths of each noise source. In Figure 80.5, the transfer function for each noise source is the same as that of the signal $v_c$. If the signal bandwidth is determined as $f_h - f_i$ by sharp filters, then

$$B_e = f_h - f_i + f_{ce} \ln \frac{f_h}{f_i} \tag{80.20}$$

$$B_{te} = B_{te} = f_h - f_i + f_{ci} \ln \frac{f_h}{f_i} \tag{80.21}$$

where $f_{ce}$ and $f_i$ are, respectively, the frequencies where the value of voltage and current noise spectral densities is twice their value at high frequency, also known as corner or 3 dB frequencies.
Another noise specification method states the peak-to-peak noise at a given low-frequency band \((f_A \text{ to } f_B)\), usually 0.1 to 10 Hz, and the noise spectral density at a frequency at which it is already constant, normally 1 or 10 kHz. In these cases, if the contribution from noise currents is negligible, the equivalent input voltage noise can be calculated from

\[
vn = \sqrt{v_{\text{nl}}^2 + v_{\text{nh}}^2}
\]  

(80.22)

where \(v_{\text{nl}}\) and \(v_{\text{nh}}\) are, respectively, the voltage noise in the low-frequency and high-frequency bands expressed in the same units (peak-to-peak or rms voltages). To convert rms voltages into peak-to-peak values, multiply by 6.6. If the signal bandwidth is from \(f_1\) to \(f_h\), and \(f_1 = f_A\) and \(f_h > f_B\), then Equation 80.22 can be written

\[
v_n = \sqrt{v_{\text{nl}}^2 + (6.6e_n)^2 (f_h - f_B)}
\]  

(80.23)

where \(v_{\text{nl}}\) is the peak-to-peak value and \(e_n\) is the rms voltage noise as specified in data books. Equation 80.23 results in a peak-to-peak calculated noise that is lower than the real noise, because noise spectral density is not constant from \(f_B\) up. However, it is a simple approach providing useful results.

For signal sources with high output resistors, thermal and excess noise from resistors (see Chapter 54) must be included. For first- and second-order filters, noise bandwidth is slightly larger than signal bandwidth. Motchenbacher and Connelly [4] show how to calculate noise bandwidth, resistor noise, and noise transfer functions when different from signal transfer functions.

Low-noise design always seeks the minimal bandwidth required for the signal. When amplifying low-frequency signals, if a large capacitor \(C_i\) is connected across the input terminals in Figure 80.5, then noise and interference having a frequency larger than \(f_0 = 1/2\pi(2R_o)C_i\) (\(f_0 << f_s\)) will be attenuated.

Another possible source of error for any IA, not included in Equation 80.5, is the *slew rate limit* of its output stage. Because of the limited current available, the voltage at the output terminal cannot change faster than a specified value \(SR\). Then, if the maximal amplitude \(A\) of an output sine wave of frequency \(f\) exceeds

\[
A = \frac{SR}{2\pi f}
\]  

(80.24)

there will be a waveform distortion.

Table 80.1 lists some basic specifications for IC instrumentation amplifies whose gain \(G\) can be set by an external resistor or a single connection.

**Instrumentation Amplifiers Built from Discrete Parts**

Instrumentation amplifiers can be built from discrete parts by using operational amplifiers (op amps) and a few resistors. An *op amp* is basically a differential voltage amplifier whose gain \(A_0\) is very large (from 10⁶ to 10⁷) at dc and rolls off (20 dB/decade) from frequencies of about 1 to 100 Hz, becoming 1 at frequencies from 1 to 10 MHz for common models (Figure 80.6a), and whose input impedances are so high (up to 10¹² \(\Omega\ || 1 \text{ pF}\)) that input currents are almost negligible. Op amps can also be modeled by the circuit in Figure 80.5, and their symbol is that in Figure 80.4a, deleting IA. However, because of their large gain, op amps cannot be used directly as amplifiers; a mere 1 mV dc input voltage would saturate any op amp output. Furthermore, op amp gain changes from unit to unit, even for the same model, and for a given unit it changes with time, temperature, and supply voltages. Nevertheless, by providing external feedback, op amps are very flexible and far cheaper than IAs. But when the cost for external components and their connections, and overall reliability are also considered, the optimal solution depends on the situation.

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<table>
<thead>
<tr>
<th>TABLE 80.1</th>
<th>Basic Specifications for Some Instrumentation Amplifiers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AD624A</td>
</tr>
<tr>
<td>Gain range</td>
<td>1–1000</td>
</tr>
<tr>
<td>Gain error, $e_G$</td>
<td></td>
</tr>
<tr>
<td>$G = 1$</td>
<td>±0.05</td>
</tr>
<tr>
<td>$G = 10$</td>
<td>n.s.</td>
</tr>
<tr>
<td>$G = 100$</td>
<td>±0.25</td>
</tr>
<tr>
<td>$G = 1000$</td>
<td>±1.0</td>
</tr>
<tr>
<td>Gain nonlinearity error $e_{NLG}$</td>
<td></td>
</tr>
<tr>
<td>$G = 1$</td>
<td>±0.005</td>
</tr>
<tr>
<td>$G = 10$</td>
<td>n.s.</td>
</tr>
<tr>
<td>$G = 100$</td>
<td>±0.005</td>
</tr>
<tr>
<td>$G = 1000$</td>
<td>±0.005</td>
</tr>
<tr>
<td>Gain drift $\Delta G/\Delta T$</td>
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</tr>
<tr>
<td>$G = 1$</td>
<td>5 ±50</td>
</tr>
<tr>
<td>$G = 10$</td>
<td>n.s.</td>
</tr>
<tr>
<td>$G = 100$</td>
<td>10 50</td>
</tr>
<tr>
<td>$G = 1000$</td>
<td>25 50</td>
</tr>
<tr>
<td>$V_m$</td>
<td>200 + 5G</td>
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<tr>
<td>$\Delta v_m/\Delta T$</td>
<td>2 + 50/G</td>
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<tr>
<td>$I_b$</td>
<td>±50 20 05</td>
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<td>±20 typ</td>
</tr>
<tr>
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</tr>
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<td>$Z_C$</td>
<td>1</td>
</tr>
<tr>
<td>CMRR at dc</td>
<td></td>
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<tr>
<td>$G = 1$</td>
<td>70 min</td>
</tr>
<tr>
<td>$G = 10$</td>
<td>n.s.</td>
</tr>
<tr>
<td>$G = 100$</td>
<td>100 min 115 min 100 min</td>
</tr>
<tr>
<td>$G = 1000$</td>
<td>110 min 115 min</td>
</tr>
<tr>
<td>PSRR at dc</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$G = 10$</td>
<td>n.s.</td>
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<tr>
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<tr>
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<td>Bandwidth (~3 dB) (typ)</td>
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<tr>
<td>$G = 1$</td>
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<tr>
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<td>Settling time to 0.01%</td>
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</tr>
<tr>
<td>$G = 10$</td>
<td>15 typ 10 typ 7.5</td>
</tr>
<tr>
<td>$G = 100$</td>
<td>15 typ 10 typ 7.5</td>
</tr>
<tr>
<td>$G = 1000$</td>
<td>75 typ 10 typ</td>
</tr>
<tr>
<td>$e_n$ (typ)</td>
<td></td>
</tr>
<tr>
<td>$G = 1$</td>
<td>4</td>
</tr>
<tr>
<td>$G = 10$</td>
<td>4</td>
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<tr>
<td>$G = 100$</td>
<td>4</td>
</tr>
<tr>
<td>$G = 1000$</td>
<td>4</td>
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<tr>
<td>$\nu_m$ 0.1 to 10 Hz (typ)</td>
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<tr>
<td>$G = 1$</td>
<td>10 10 1</td>
</tr>
<tr>
<td>$G = 10$</td>
<td>n.s.</td>
</tr>
<tr>
<td>$G = 100$</td>
<td>0.3 0.5 1</td>
</tr>
<tr>
<td>$G = 1000$</td>
<td>0.2 0.4 1</td>
</tr>
</tbody>
</table>

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Figure 80.6b shows an amplifier built from an op amp with external feedback. If input currents are neglected, the current through $R_2$ will flow through $R_1$ and we have
\begin{equation}
v_d = v_o - v_s \frac{R_1}{R_1 + R_2} \tag{80.25}
\end{equation}
\begin{equation}
v_o = A_d v_d \tag{80.26}
\end{equation}
Therefore,
\begin{equation}
v_o = A_d \left(1 + \frac{R_2}{R_1}\right) \frac{G_i}{A_d} = \frac{G_i}{1 + \frac{G_i}{A_d}} \tag{80.27}
\end{equation}
where $G_i = 1 + R_2/R_1$ is the ideal gain for the amplifier. If $G_i/A_d$ is small enough ($G_i$ small, $A_d$ large), the gain does not depend on $A_d$ but only on external components. At high frequencies, however, $A_d$ becomes smaller and, from Equation 80.27, $v_o < G_i v_s$ so that the bandwidth for the amplifier will reduce for large gains. Franco [5] analyzes different op amp circuits useful for signal conditioning.

Figure 80.7 shows an IA built from three op amps. The input stage is fully differential and the output stage is a difference amplifier converting a differential voltage into a single-ended output voltage. Difference amplifiers (op amp and matched resistors) are available in IC form: AMP 03 (Analog Devices) and INA 105/6 and INA 117 (Burr-Brown). The gain equation for the complete IA is
\begin{equation}
G = \left(1 + 2 \frac{R_2}{R_1}\right) \frac{R_2}{R_1} \tag{80.28}
\end{equation}
Pallás-Areny and Webster [6] have analyzed matching conditions in order to achieve a high CMRR. Resistors $R_2$ do not need to be matched. Resistors $R_3$ and $R_4$ need to be closely matched. A potentiometer connected to the $v_{ref}$ terminal makes it possible to trim the CMRR at low frequencies.

The three-op-amp IA has a symmetrical structure making it easy to design and test. IAs based on an IC difference amplifier do not need any user trim for high CMRR. The circuit in Figure 80.8 is an IA that lacks these advantages but uses only two op amps. Its gain equation is

\begin{table}[h]
\centering
\caption{Basic Specifications for Some Instrumentation Amplifiers (continued)}
\begin{tabular}{|c|c|c|c|c|}
\hline
& AD624A & AMP02F & INA110KP & LT1101AC & Units \\
\hline
\hline
$i_s$ (0.1 to 10 Hz) (typ) & 60 & n.s. & n.s. & 2.3 & pA-p-p \\
$i_s$ (typ) & n.s. & 400 & 1.8 & 20 & pA/\sqrt{Hz} \\
\hline
\end{tabular}
\end{table}

Note: All parameter values are maximum, unless otherwise stated (typ = typical; min = minimum; n.a. = not applicable; n.s. = not specified). Measurement conditions are similar; consult manufacturers’ data books for further detail.

\* For the INA110, the gain nonlinearity error is specified as percentage of the full-scale output.

\*\* Input current drift for the INA110KP approximately doubles for every 10°C increase, from 25°C (10 pA-typ) to 125°C (10 nA-typ).

\*\*\* The PSRR for the INA110 is specified as an input offset $\pm(10 + 180/G)$ $\mu$V/V maximum.

\begin{itemize}
\item Note: All parameter values are maximum, unless otherwise stated (typ = typical; min = minimum; n.a. = not applicable; n.s. = not specified). Measurement conditions are similar; consult manufacturers’ data books for further detail.
\item For the INA110, the gain nonlinearity error is specified as percentage of the full-scale output.
\item Input current drift for the INA110KP approximately doubles for every 10°C increase, from 25°C (10 pA-typ) to 125°C (10 nA-typ).
\item The PSRR for the INA110 is specified as an input offset $\pm(10 + 180/G)$ $\mu$V/V maximum.
\end{itemize}
FIGURE 80.6  (a) Open loop gain for an op amp. (b) Amplifier based on an op amp with external feedback.
FIGURE 80.7 Instrumentation amplifier built from three op amps. $R_3$ and $R_4$ must be matched.

FIGURE 80.8 Instrumentation amplifier built from two op amps. $R_1$ and $R_2$ must be matched.
80.12 Special-Purpose Signal Conditioners

Table 80.4 lists some signal conditioners in IC form intended for specific sensors and describes their respective functions. The decision whether to design a signal conditioner from parts or use a model from Table 80.4 is a matter of cost, reliability, and availability. Signal conditioners are also available as subsystems (plug-in cards and modules), for example, series MB from Keithley Metrabyte, SCM from Burr-Brown, and 3B, 5B, 6B, and 7B from Analog Devices.

### Defining Terms

**Carrier amplifier:** Voltage amplifier for narrowband ac signals, that includes in addition a sine wave oscillator, a synchronous demodulator, and a low-pass filter.

**Common-mode rejection ratio (CMRR):** The gain for a differential voltage divided by the gain for a common-mode voltage in a differential amplifier. It is usually expressed in decibels.

**Common-mode voltage:** The average of the voltages at the input terminals of a differential amplifier.

**Differential amplifier:** Circuit or device that amplifies the difference in voltage between two terminals, none of which is grounded.

**Dynamic range:** The measurement range for a quantity divided by the desired resolution.

**Instrumentation amplifier:** Differential amplifier with large input impedance and low offset and gain errors.

**Isolation amplifier:** Voltage amplifier whose ground terminal for input voltages is independent from the ground terminal for the output voltage (i.e., there is a large impedance between both ground terminals).

**Isolation Mode Rejection Ratio (IMRR):** The amplitude of the output voltage of an isolation amplifier divided by the voltage across the isolation impedance yielding that voltage.

**Signal conditioner:** Circuit or device that adapts a sensor signal to an ensuing circuit, such as an analog-to-digital converter.

**Voltage buffer:** Voltage amplifier whose gain is 1, or close to 1, and whose input impedance is very large while its output impedance is very small.

<table>
<thead>
<tr>
<th>Model</th>
<th>Function</th>
<th>Manufacturer</th>
</tr>
</thead>
<tbody>
<tr>
<td>4341</td>
<td>rms-to-dc converter</td>
<td>Burr-Brown</td>
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<tr>
<td>ACF2101</td>
<td>Low-noise switched integrator</td>
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<td>AD1860</td>
<td>Intelligent digitizing signal conditioner</td>
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<tr>
<td>AD2953</td>
<td>LVDT-to-digital converter (ac bridge conditioner)</td>
<td>Analog Devices</td>
</tr>
<tr>
<td>AD594</td>
<td>Thermocouple amplifier with cold junction compensation</td>
<td>Analog Devices</td>
</tr>
<tr>
<td>AD596/7</td>
<td>Thermocouple conditioner and set-point controllers</td>
<td>Analog Devices</td>
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<tr>
<td>AD598</td>
<td>LVDT signal conditioner</td>
<td>Analog Devices</td>
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<td>rms-to-dc (rms-to-dc converter)</td>
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<td>Signal conditioning ADC</td>
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<td>AD698</td>
<td>LVDT signal conditioner</td>
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<td>Signal conditioning ADC with RTD excitation currents</td>
<td>Analog Devices</td>
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<tr>
<td>AD7711</td>
<td>Signal conditioning ADC with RTD excitation currents</td>
<td>Analog Devices</td>
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<tr>
<td>IMP50E10</td>
<td>Electrically programmable analog circuit</td>
<td>IMP</td>
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<td>Fluid level detector</td>
<td>National Semiconductor</td>
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<td>LM1042</td>
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<td>LT9001</td>
<td>Thermocouple cold junction compensator and matched amplifier</td>
<td>Linear Technology</td>
</tr>
<tr>
<td>TLE2425</td>
<td>Precision virtual ground</td>
<td>Texas Instruments</td>
</tr>
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</table>
References


Further Information


M.N. Horenstein, *Microelectronic Circuits and Devices*, 2nd ed., Englewood Cliffs, NJ: Prentice-Hall, 1996, is an introductory electronics textbook for electrical or computer engineering students. It provides many examples and proposes many more problems, for some of which solutions are offered.


T.H. Wilmshurst, *Signal Recovery from Noise in Electronic Instrumentation*, 2nd ed., Bristol, U.K.: Adam Hilger, 1990, describes various techniques for reducing noise and interference in instrumentation. No references are provided and some demonstrations are rather short, but it provides insight into very interesting topics.

Manufacturers’ data books provide a wealth of information, albeit nonuniformly. Application notes for special components should be consulted before undertaking any serious project. In addition, application notes provide handy solutions to difficult problems and often inspire good designs. Most manufacturers offer such literature free of charge. The following have shown to be particularly useful and easy to obtain: 1993 Applications Reference Manual, Analog Devices; 1994 IC Applications Handbook, Burr-Brown; 1990 Linear Applications Handbook and 1993 Linear Applications Handbook Vol. II, Linear Technology; 1994 Linear Application Handbook, National Semiconductor; Linear and Interface Circuit Applications, Vols. 1, 2, and 3, Texas Instruments.

82.1 Introduction

In its broadest sense, a filter can be defined as a signal processing system whose output signal, usually called the response, differs from the input signal, called the excitation, such that the output signal has some prescribed properties. In more practical terms an electric filter is a device designed to suppress, pass, or separate a group of signals from a mixture of signals according to the specifications in a particular application. The application areas of filtering are manifold, for example to band-limit signals before sampling to reduce aliasing, to eliminate unwanted noise in communication systems, to resolve signals into their frequency components, to convert discrete-time signals into continuous-time signals, to demodulate signals, etc. Filters are generally classified into three broad classes: continuous-time, sampled-data, and discrete-time filters depending on the type of signal being processed by the filter. Therefore, the concept of signals are fundamental in the design of filters.

A signal is a function of one or more independent variables such as time, space, temperature, etc. that carries information. The independent variables of a signal can either be continuous or discrete. Assuming that the signal is a function of time, in the first case the signal is called continuous-time and in the second, discrete-time. A continuous-time signal is defined at every instant of time over a given interval, whereas a discrete-time signal is defined only at a discrete-time instances. Similarly, the values of a signal can also be classified in either continuous or discrete.
In real-world signals, often referred to as analog signals, both amplitude and time are continuous. These types of signals cannot be processed by digital machines unless they have been converted into discrete-time signals. By contrast, a digital signal is characterized by discrete signal values, that are defined only at discrete points in time. Digital signal values are represented by a finite number of digits, which are usually binary coded. The relationship between a continuous-time signal and the corresponding discrete-time signal can be expressed in the following form:

\[ x(kT) = x\left(\frac{t}{T}\right), \quad k = 0, 1, 2, \ldots, \]  

where \( T \) is called the sampling period.

Filters can be classified on the basis of the input, output, and internal operating signals. A continuous data filter is used to process continuous-time or analog signals, whereas a digital filter processes digital signals. Continuous data filters are further divided into passive or active filters, depending on the type of elements used in their implementation. Perhaps the earliest type of filters known in the engineering community are LC filters, which can be designed by using discrete components like inductors and capacitors, or crystal and mechanical filters that can be implemented using LC equivalent circuits. Since no external power is required to operate these filters, they are often referred to as passive filters. In contrast, active filters are based on active devices, primarily \( RC \) elements, and amplifiers. In a sampled data filter, on the other hand, the signal is sampled and processed at discrete instants of time. Depending on the type of signal processed by such a filter, one may distinguish between an analog sampled data filter and a digital filter. In an analog sampled data filter the sampled signal can principally take any value, whereas in a digital filter the sampled signal is a digital signal, the definition of which was given earlier. Examples of analog sampled data filters are switched capacitor (SC) filters and charge-transfer device (CTD) filters made of capacitors, switches, and operational amplifiers.

### 82.2 Filter Classification

Filters are commonly classified according to the filter function they perform. The basic functions are: low-pass, high-pass, bandpass, and bandstop. If a filter passes frequencies from zero to its cutoff frequency \( \Omega_c \) and stops all frequencies higher than the cutoff frequencies, then this filter type is called an ideal low-pass filter. In contrast, an ideal high-pass filter stops all frequencies below its cutoff frequency and passes all frequencies above it. Frequencies extending from \( \Omega_1 \) to \( \Omega_2 \) are passed by an ideal bandpass filter, while all other frequencies are stopped. An ideal bandstop filter stops frequencies from \( \Omega_1 \) to \( \Omega_2 \) and passes all other frequencies. Figure 82.1 depicts the magnitude functions of the four basic ideal filter types.

So far we have discussed ideal filter characteristics having rectangular magnitude responses. These characteristics, however, are physically not realizable. As a consequence, the ideal response can only be approximated by some nonideal realizable system. Several classical approximation schemes have been developed, each of which satisfies a different criterion of optimization. This should be taken into account when comparing the performance of these filter characteristics.

### 82.3 The Filter Approximation Problem

Generally the input and output variables of a linear, time-invariant, causal filter can be characterized either in the time-domain through the convolution integral given by

\[ y(t) = \int_{0}^{\infty} h_a(t - \tau) x(\tau) d\tau \]  

or, equivalently, in the frequency-domain through the transfer function
where $H_a(s)$ is the Laplace transform of the impulse response $h_a(t)$ and $X(s)$, $Y(s)$ are the Laplace transforms of the input signal $x(t)$ and the output or the filtered signal $y(t)$. $X(s)$ and $Y(s)$ are polynomials in $s = \sigma + j\Omega$ and the overall transfer function $H_a(s)$ is a real rational function of $s$ with real coefficients. The zeroes of the polynomial $X(s)$ given by $s = s_i$ are called the poles of $H_a(s)$ and are commonly referred to as the natural frequencies of the filter. The zeros of $Y(s)$ given by $s = s_0i$ which are equivalent to the zeroes of $H_a(s)$ are called the transmission zeros of the filter. Clearly, at these frequencies the filter output is zero for any finite input. Stability restricts the poles of $H_a(s)$ to lie in the left half of the $s$-plane excluding the $j\Omega$-axis, that is $\text{Re}\{s_i\} < 0$. For a stable transfer function $H_a(s)$ reduces to $H_a(j\Omega)$ on the $j\Omega$-axis, which is the continuous-time Fourier transform of the impulse response $h_a(t)$ and can be expressed in the following form:

$$H_a(s) = \frac{Y(s)}{X(s)} = \sum_{i=0}^{N} b_i s^i \quad \Leftrightarrow \quad H_a(s) = \frac{b_{N}}{a_N} \prod_{i=1}^{N} \left(\frac{s-s_i}{s-s_0i}\right)$$  \hspace{1cm} (82.3)

where $H_a(s)$ is the Laplace transform of the impulse response $h_a(t)$ and $X(s)$, $Y(s)$ are the Laplace transforms of the input signal $x(t)$ and the output or the filtered signal $y(t)$. $X(s)$ and $Y(s)$ are polynomials in $s = \sigma + j\Omega$ and the overall transfer function $H_a(s)$ is a real rational function of $s$ with real coefficients. The zeroes of the polynomial $X(s)$ given by $s = s_i$ are called the poles of $H_a(s)$ and are commonly referred to as the natural frequencies of the filter. The zeros of $Y(s)$ given by $s = s_0i$ which are equivalent to the zeroes of $H_a(s)$ are called the transmission zeros of the filter. Clearly, at these frequencies the filter output is zero for any finite input. Stability restricts the poles of $H_a(s)$ to lie in the left half of the $s$-plane excluding the $j\Omega$-axis, that is $\text{Re}\{s_i\} < 0$. For a stable transfer function $H_a(s)$ reduces to $H_a(j\Omega)$ on the $j\Omega$-axis, which is the continuous-time Fourier transform of the impulse response $h_a(t)$ and can be expressed in the following form:

$$H_a(j\Omega) = |H_a(j\Omega)| e^{j\theta(\Omega)}$$  \hspace{1cm} (82.4)

where $|H_a(j\Omega)|$ is called the magnitude function and $\theta(\Omega) = \text{arg} H_a(j\Omega)$ is the phase function. The gain magnitude of the filter expressed in decibels (dB) is defined by

$$\alpha(\Omega) = 20 \log |H_a(j\Omega)| = 10 \log |H_a(j\Omega)|^2$$  \hspace{1cm} (82.5)

Note that a filter specification is often given in terms of its attenuation, which is the negative of the gain function also given in decibels. While the specifications for a desired filter behavior are commonly given in terms of the loss response $\alpha(\Omega)$, the solution of the filter approximation problem is always carried out with the help of the characteristic function $C(j\Omega)$ giving

$$\alpha(\Omega) = 10 \log \left[ 1 + |C(j\Omega)|^2 \right]$$  \hspace{1cm} (82.6)
Note that $\alpha(\Omega)$ is not a rational function, but $C(j\Omega)$ can be a polynomial or a rational function and approximation with polynomial or rational functions is relatively convenient. It can also be shown that frequency-dependent properties of $|C(j\Omega)|$ are in many ways identical to those of $\alpha(\Omega)$. The approximation problem consists of determining a desired response $|H_a(j\Omega)|$ such that the typical specifications depicted in Figure 82.2 are met. This so-called tolerance scheme is characterized by the following parameters:

- $\Omega_p$: Passband cutoff frequency (rad/s)
- $\Omega_s$: Stopband cutoff frequency (rad/s)
- $\Omega_c$: $-3$ dB cutoff frequency (rad/s)
- $\varepsilon$: Permissible error in passband given by $\varepsilon = (10^{\alpha_{dB}} - 1)^{1/2}$, where $\alpha$ is the maximum acceptable attenuation in dB; note that $10 \log (1 + \varepsilon^2)^{1/2} = -\alpha$
- $1/A$: Permissible maximum magnitude in the stopband, i.e., $A = 10^{\alpha_{dB}}$, where $\alpha$ is the minimum acceptable attenuation in dB; note that $20 \log (1/A) = -\alpha$.

The **passband** of a low-pass filter is the region in the interval $[0, \Omega_p]$ where the desired characteristics of a given signal are preserved. In contrast, the **stopband** of a low-pass filter (the region $[\Omega_s, \infty]$) rejects signal components. The **transition band** is the region between $(\Omega_c - \Omega_p)$, which would be 0 for an ideal filter. Usually, the amplitudes of the permissible ripples for the magnitude response are given in decibels.

The following sections review four different classical approximations: Butterworth, Chebyshev Type I, elliptic, and Bessel.

### Butterworth Filters

The frequency response of an $N$th-order Butterworth low-pass filter is defined by the squared magnitude function

$$
|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}
$$

It is evident from the Equation 82.7 that the Butterworth approximation has only poles, i.e., no finite zeros and yields a maximally flat response around zero and infinity. Therefore, this approximation is also...
called maximally flat magnitude (MFM). In addition, it exhibits a smooth response at all frequencies
and a monotonic decrease from the specified cutoff frequencies.

Equation 82.7 can be extended to the complex s-domain, resulting in

\[ H_s(s)H_s(-s) = \frac{1}{1+(s/jΩc)^{2N}} \]  \hspace{1cm} (82.8)

The poles of this function are given by the roots of the denominator

\[ s_k = Ωc e^{jπ(2k+1)/4N}, \quad k = 0, 1, \ldots, 2N-1 \] \hspace{1cm} (82.9)

Note that for any N, these poles lie on the unit circle of radius Ωc in the s-plane. To guarantee stability,
the poles that lie in the left half-plane are identified with \( H_s(s) \). As an example, we will determine the
transfer function corresponding to a third-order Butterworth filter, i.e., \( N = 3 \).

\[ H_s(s)H_s(-s) = \frac{1}{1+(-s)^3} = \frac{1}{1-s^3} \]  \hspace{1cm} (82.10)

The roots of denominator of Equation 82.10 are given by

\[ s_k = Ωe^{jπ(2k+1)/6}, \quad k = 0, 1, 2, 3, 4, 5 \] \hspace{1cm} (82.11)

Therefore, we obtain

\[ s_0 = Ωe^{jπ/3} = -1/2 + j\sqrt{3}/2 \]
\[ s_1 = Ωe^{jπ} = -1 \]
\[ s_2 = Ωe^{jπ/3} = -1/2 - j\sqrt{3}/2 \]
\[ s_3 = Ωe^{jπ/3} = 1/2 - j\sqrt{3}/2 \]
\[ s_4 = Ωe^{j2π} = 1 \]
\[ s_5 = Ωe^{jπ/3} = 1/2 + j\sqrt{3}/2 \] \hspace{1cm} (82.12)

The corresponding transfer function is obtained by identifying the left half-plane poles with \( H_s(s) \). Note
that for the sake of simplicity we have chosen \( Ωc = 1 \).

\[ H(s) = \frac{1}{(s+1)(s+1/2-j\sqrt{3}/2)(s+1/2+j\sqrt{3}/2)} = \frac{1}{1+2s+2s^2+s^3} \] \hspace{1cm} (82.13)

Table 82.1 gives the Butterworth denominator polynomials up \( N = 5 \).

Table 82.2 gives the Butterworth poles in real and imaginary components and in frequency and Q.
In the next example, the order of a low-pass Butterworth filter is to be determined whose cutoff frequency (–3 dB) is \( W_c = 2 \text{ kHz} \) and stopband attenuation is greater than 40 dB at \( W_s = 6 \text{ kHz} \). Thus the desired filter specification is

\[
20 \log |H_{s}(j\Omega)| \leq -40, \quad \Omega \geq W_c
\]  

(82.14)

or equivalently,

\[
|H_{s}(j\Omega)| \leq 0.01, \quad \Omega \geq W_c
\]  

(82.15)

It follows from Equation 82.7

\[
\frac{1}{1 + \left(\frac{W_s}{W_c}\right)^{2N}} = \left(0.01\right)^{2}
\]  

(82.16)

<table>
<thead>
<tr>
<th>TABLE 82.1</th>
<th>Butterworth Denominator Polynomials</th>
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<tbody>
<tr>
<td>Order ( N )</td>
<td>Butterworth Denominator Polynomials of ( H(s) )</td>
</tr>
<tr>
<td>1</td>
<td>( s + 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( s^2 + s + 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( s^3 + 2s^2 + 2s + 1 )</td>
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<tr>
<td>4</td>
<td>( s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1 )</td>
</tr>
<tr>
<td>5</td>
<td>( s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1 )</td>
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<table>
<thead>
<tr>
<th>TABLE 82.2</th>
<th>Butterworth and Bessel Poles</th>
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<tr>
<td>Butterworth Poles</td>
<td>Bessel Poles (-3 dB)</td>
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<tr>
<td>( N )</td>
<td>Re</td>
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<tr>
<td>1</td>
<td>-1.000</td>
</tr>
<tr>
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<td>-0.707</td>
</tr>
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</table>

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Solving the above equation for \( N \) gives \( N = 4.19 \). Since \( N \) must be an integer, a fifth-order filter is required for this specification.

### Chebyshev Filters or Chebyshev I Filters

The frequency response of an \( N \)th-order Chebyshev low-pass filter is specified by the squared-magnitude frequency response function

\[
\left| H_a(j\Omega) \right|^2 = \frac{1}{1 + \varepsilon^2 T_N^2(\Omega_p/\Omega_p)} \tag{82.17}
\]

where \( T_N(x) \) is the \( N \)th-order Chebyshev polynomial and \( \varepsilon \) is a real constant less than 1 which determines the ripple of the filter. Specifically, for nonnegative integers \( N \), the \( N \)th-order Chebyshev polynomial is given by

\[
T_N(x) = \begin{cases} 
\cos\left( N \cos^{-1} x \right), & x \leq 1 \\
\cosh\left( N \cosh^{-1} x \right), & x \geq 1 
\end{cases} \tag{82.18}
\]

High-order Chebyshev polynomials can be derived from the recursion relation

\[
T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x) \tag{82.19}
\]

where \( T_0(x) = 1 \) and \( T_1(x) = x \).

The Chebyshev approximation gives an equiripple characteristic in the passband and is maximally flat near infinity in the stopband. Each of the Chebyshev polynomials has real zeros that lie within the interval \((-1,1)\) and the function values for \( x \in [-1,1] \) do not exceed +1 and -1.

The pole locations for Chebyshev filter can be determined by generating the appropriate Chebyshev polynomials, inserting them into Equation 82.17, factoring, and then selecting only the left half plane roots. Alternatively, the pole locations \( P_k \) of an \( N \)th-order Chebyshev filter can be computed from the relation, for \( k = 1 \to N \)

\[
P_k = -\sin \Theta_k \sinh \beta + j \cos \Theta_k \cosh \beta \tag{82.20}
\]

where \( \Theta_k = (2k - 1)\pi/2N \) and \( \beta = \sinh^{-1}(1/\varepsilon) \).

**Note:** \( P_{N-k+1} \) and \( P_k \) are complex conjugates and when \( N \) is odd there is one real pole at

\[
P_{N+1} = -2 \sinh \beta
\]

For the Chebyshev polynomials, \( \Omega_p \) is the last frequency where the amplitude response passes through the value of ripple at the edge of the passband. For odd \( N \) polynomials, where the ripple of the Chebyshev polynomial is negative going, it is the \((-1/(1 + \varepsilon^2))^{1/2}) \) frequency and for even \( N \), where the ripple is positive going, is the 0 dB frequency.

The Chebyshev filter is completely specified by the three parameters \( \varepsilon, \Omega_p, \) and \( N \). In a practical design application, \( \varepsilon \) is given by the permissible passband ripple and \( \Omega_p \) is specified by the desired passband cutoff frequency. The order of the filter, i.e., \( N \), is then chosen such that the stopband specifications are satisfied.
Elliptic or Cauer Filters

The frequency response of an \( N \)th-order elliptic low-pass filter can be expressed by

\[
|H_e(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 F_N^2(\Omega/\Omega_p)}
\]  

(82.21)

where \( F_N(\cdot) \) is called the Jacobian elliptic function. The elliptic approximation yields an equiripple passband and an equiripple stopband. Compared with the same-order Butterworth or Chebyshev filters, the elliptic design provides the sharpest transition between the passband and the stopband. The theory of elliptic filters, initially developed by Cauer, is involved, therefore for an extensive treatment refer to Reference 1.

Elliptic filters are completely specified by the parameters \( \epsilon, \alpha, \Omega_p, \Omega_s, \) and \( N \)

where

- \( \epsilon \) = passband ripple
- \( \alpha \) = stop band floor
- \( \Omega_p \) = the frequency at the edge of the passband (for a designated passband ripple)
- \( \Omega_s \) = the frequency at the edge of the stopband (for a designated stopband floor)
- \( N \) = the order of the polynomial

In a practical design exercise, the desired passband ripple, stopband floor, and \( \Omega \) are selected and \( N \) is determined and rounded up to the nearest integer value. The appropriate Jacobian elliptic function must be selected and \( H_e(j\Omega) \) must be calculated and factored to extract only the left plane poles. For some synthesis techniques, the roots must expanded into polynomial form.

This process is a formidable task. While some filter manufacturers have written their own computer programs to carry out these calculations, they are not readily available. However, the majority of applications can be accommodated by use of published tables of the pole/zero configurations of low-pass elliptic transfer functions. An extensive set of such tables for a common selection of passband ripples, stopband floors, and shape factors is available in Reference 2.

Bessel Filters

The primary objectives of the preceding three approximations were to achieve specific loss characteristics. The phase characteristics of these filters, however, are nonlinear. The Bessel filter is optimized to reduce nonlinear phase distortion, i.e., a maximally flat delay. The transfer function of a Bessel filter is given by

\[
H_b(s) = \frac{B_0}{B_n(s)} = \frac{B_0}{\sum_{k=0}^{N} B_k s^k}, \quad B_k = \frac{(2N-k)!}{2^{N-k} k!(N-k)!} \quad k = 0, 1, \ldots, N
\]  

(82.22)

where \( B_n(s) \) is the \( N \)th-order Bessel polynomial. The overall squared-magnitude frequency response function is given by

\[
|H_b(j\Omega)|^2 = 1 - \frac{\Omega^2}{2N-1} + \frac{2(N-1)\Omega^4}{(2N-1)^2(2N-3)} + \cdots
\]  

(82.23)

To illustrate Equation 82.22 the Bessel transfer function for \( N = 4 \) is given below:

\[
H_b(s) = \frac{105}{105 + 105s + 45s^2 + 10s^3 + s^4}
\]  

(82.24)
Table 82.2 lists the factored pole frequencies as real and imaginary parts and as frequency and $Q$ for Bessel transfer functions that have been normalized to $\Omega_c = -3\,\text{dB}$.

### 82.4 Design Examples for Passive and Active Filters

#### Passive $R$, $L$, $C$ Filter Design

The simplest and most commonly used passive filter is the simple, first-order ($N = 1$) $R$–$C$ filter shown in Figure 82.3. Its transfer function is that of a first-order Butterworth low-pass filter. The transfer function and $-3\,\text{dB}$ $\Omega_c$ are

$$H(s) = \frac{1}{RCS + 1} \quad \text{where} \quad \Omega_c = \frac{1}{RC} \quad (82.25)$$

While this is the simplest possible filter implementation, both source and load impedance change the dc gain and/or corner frequency and its rolloff rate is only first order, or $-6\,\text{dB/octave}$.

To realize higher-order transfer functions, passive filters use $R$, $L$, $C$ elements usually configured in a ladder network. The design process is generally carried out in terms of a doubly terminated two-port network with source and load resistors $R_1$ and $R_2$ as shown in Figure 82.4. Its symbolic representation is given below.

The source and load resistors are normalized in regard to a reference resistance $R_B = R_1$, i.e.,

$$r_1 = \frac{R_1}{R_B} = 1, \quad r_2 = \frac{R_2}{R_B} = \frac{R_2}{R_1} \quad (82.26)$$

The values of $L$ and $C$ are also normalized in respect to a reference frequency to simplify calculations. Their values can be easily scaled to any desired set of actual elements.

$$l_v = \frac{\Omega_B L_v}{R_B}, \quad c_v = \frac{\Omega_B C_v R_B}{} \quad (82.27)$$

**FIGURE 82.3** A passive first-order $RC$ filter can serve as an antialiasing filter or to minimize high-frequency noise.

**FIGURE 82.4** A passive filter can have the symbolic representation of a doubly terminated filter.
Low-pass filters, whose magnitude-squared functions have no finite zero, i.e., whose characteristic functions \( C(j\Omega) \) are polynomials, can be realized by lossless ladder networks consisting of inductors as the series elements and capacitors as the shunt elements. These types of approximations, also referred to as all-pole approximations, include the previously discussed Butterworth, Chebyshev Type I, and Bessel filters. Figure 82.5 shows four possible ladder structures for even and odd \( N \), where \( N \) is the filter order.

In the case of doubly terminated Butterworth filters, the normalized values are precisely given by

\[
a_v = 2 \sin \left( \frac{2v-1}{2N} \pi \right), \quad v = 1, \ldots, N
\]  

(82.28)

where \( a_v \) is the normalized \( L \) or \( C \) element value. As an example we will derive two possible circuits for a doubly terminated Butterworth low-pass of order 3 with \( R_B = 100 \Omega \) and a cutoff frequency \( \Omega_c = \Omega_B = 10 \) kHz. The element values from Equation 82.28 are

\[
\begin{align*}
L_1 &= 2 \sin \left( \frac{2-1}{6} \pi \right) = 1 \Rightarrow L_1 = \frac{R_B}{\Omega_c} = 1.59 \text{ mH} \\
C_2 &= 2 \sin \left( \frac{4-1}{6} \pi \right) = 2 \Rightarrow C_2 = \frac{2}{\Omega_c R_B} = 3.183 \text{ nF} \quad \text{(82.29)} \\
L_3 &= 2 \sin \left( \frac{6-1}{6} \pi \right) = 1 \Rightarrow L_3 = \frac{R_B}{\Omega_c} = 1.59 \text{ mH}
\end{align*}
\]

A possible realization is shown in Figure 82.6.

A third-order passive all-pole filter can be realized by a doubly terminated third-order circuit.
Table 82.3 gives normalized element values for the various all-pole filter approximations discussed in the previous section up to order 3 and is based on the following normalization:

1. \( r_1 = 1 \);
2. All the cutoff frequencies (end of the ripple band for the Chebyshev approximation) are \( \Omega_c = 1 \) rad/s;
3. \( r_2 \) is either 1 or \( \infty \), so that both singly and doubly terminated filters are included.

The element values in Table 82.3 are numbered from the source end in the same manner as in Figure 82.4. In addition, empty spaces indicate unrealizable networks. In the case of the Chebyshev filter, the amount of ripple can be specified as desired, so that in the table only a selective sample can be given. Extensive tables of prototype element values for many types of filters can be found in Reference 4.

The example given above, of a Butterworth filter of order 3, can also be verified using Table 82.3. The steps necessary to convert the normalized element values in the table into actual filter values are the same as previously illustrated.

In contrast to all-pole approximations, the characteristic function of an elliptic filter function is a rational function. The resulting filter will again be a ladder network but the series elements may be parallel combinations of capacitance and inductance and the shunt elements may be series combinations of capacitance and inductance.

Figure 82.5 illustrates the general circuits for even and odd \( N \), respectively. As in the case of all-pole approximations, tabulations of element values for normalized low-pass filters based on elliptic approximations are also possible. Since these tables are quite involved the reader is referred to Reference 4.

### Active Filter Design

Active filters are widely used and commercially available with cutoff frequencies from millihertz to megahertz. The characteristics that make them the implementation of choice for several applications are small size for low frequency filters because they do not use inductors; precision realization of theoretical transfer functions by use of precision resistors and capacitors; high input impedance that is easy to drive and for many circuit configurations the source impedance does not effect the transfer function; low output impedance that can drive loads without effecting the transfer function and can drive the transient, switched capacitive, loads of the input stages of A/D converters and low (N+THD) performance for pre-A/D antialiasing applications (as low as –100 dBc).

Active filters use \( R, C, A \) (operational amplifier) circuits to implement polynomial transfer functions. They are most often configured by cascading an appropriate number of first- and second-order sections.

The simplest first-order (\( N = 1 \)) active filter is the first-order passive filter of Figure 82.3 with the addition of a unity gain follower amplifier. Its cutoff frequency (\( \Omega_c \)) is the same as that given in Equation 82.25. Its advantage over its passive counterpart is that its operational amplifier can drive whatever load that it can tolerate without interfering with the transfer function of the filter.
The vast majority of higher-order filters have poles that are not located on the negative real axis in the s-plane and therefore are in complex conjugate pairs that combine to create second-order pole pairs of the form:

\[ H(s) = \frac{s^2 + \omega_p^2}{Q} s + \frac{\omega_p^2}{Q} \leftrightarrow s^2 + \frac{2a}{Q} s + a^2 + b^2 \]  

(82.30)

where \( p_1, p_2 = a \pm jb \)

\[ \omega_p = a^2 + b^2 \]

\[ Q = \frac{\omega_p}{2a} = \frac{\sqrt{(a^2 + b^2)}}{2a} \]

The most commonly used two-pole active filter circuits are the Sallen and Key low-pass resonator, the multiple feedback bandpass, and the state variable implementation as shown in Figure 82.7a, b, and c. In the analyses that follow, the more commonly used circuits are used in their simplest form. A more comprehensive treatment of these and numerous other circuits can be found in Reference 20.

The Sallen and Key circuit of Figure 82.7a is used primarily for its simplicity. Its component count is the minimum possible for a two-pole active filter. It cannot generate stopband zeros and therefore is limited in its use to monotonic roll-off transfer functions such as Butterworth and Bessel filters. Other limitations are that the phase shift of the amplifier reduces the Q of the section and the capacitor ratio becomes large for high-Q circuits. The amplifier is used in a follower configuration and therefore is subjected to a large common mode input signal swing which is not the best condition for low distortion performance. It is recommended to use this circuit for a section \( Q < 10 \) and to use an amplifier whose gain bandwidth product is greater than \( 100 f_p \).

The transfer function and design equations for the Sallen and Key circuit of Figure 82.7a are

\[ H(s) = \frac{1}{R_2C_2} s + \frac{1}{R_1C_1} \leftrightarrow s^2 + \frac{\omega_p^2}{Q} s + \frac{\omega_p^2}{Q} \]  

(82.31)
from which obtains

$$\omega^2 = \frac{1}{R_1 R_2 C_1 C_2}, \quad Q = \omega_p R_1 C_2 = \sqrt{\frac{R_2}{R_1 C_1}} \quad (82.32)$$

$$R_1, R_2 = \frac{1}{4\pi f_p Q C_2} \left[ 1 \pm \sqrt{1 - \frac{4Q^2 C_2}{C_1}} \right] \quad (82.33)$$

which has valid solutions for

$$\frac{C_1}{C_2} \geq 4Q^2 \quad (82.34)$$

In the special case where

$$R_1 = R_2 = R, \quad \text{then}$$

$$C_1 = \frac{1}{2\pi f_p R}, \quad C_2 = Q C_1 \quad \text{and} \quad C_2 = C/2Q \quad (82.35)$$

The design sequence for Sallen and Key low-pass of Figure 82.7a is as follows:

For a required \( f_p \) and \( Q \), select \( C_1, C_2 \) to satisfy Equation 82.34. Compute \( R_1, R_2 \) from Equation 82.33 (or Equation 82.35 if \( R_1 \) is chosen to equal \( R_2 \)) and scale the values of \( C_1 \) and \( C_2 \) and \( R_1 \) and \( R_2 \) to desired impedance levels.

As an example, a three-pole low-pass active filter is shown in Figure 82.8. It is realized with a buffered single-pole \( RC \) low-pass filter section in cascade with a two-pole Sallen and Key section.

To construct a three-pole Butterworth filter, the pole locations are found in Table 82.2 and the element values in the sections are calculated from Equation 82.25 for the single real pole and in accordance with the Sallen and Key design sequence listed above for the complex pole pair.

From Table 82.2, the normalized pole locations are

$$f_{p1} = 1.000, \quad f_{p2} = 1.000, \quad \text{and} \quad Q_{p2} = 1.000$$

For a cutoff frequency of 10 kHz and if it is desired to have an impedance level of 10 k\( \Omega \), then the capacitor values are computed as follows:
For $R = 10 \, k\Omega$:

from Equation 82.25, $C_1 = \frac{1}{2\pi R f_p^1} = \frac{1}{2\pi \left(10,000\right) \left(10,000\right)} = \frac{10^{-6}}{200\pi} = 0.00159 \, \mu F$

For $R_2 = R_3 = R = 10 \, k\Omega$:

from Equation 82.35, $C = \frac{1}{2\pi R f_p^2} = \frac{1}{2\pi \left(10,000\right) \left(10,000\right)} = \frac{10^{-6}}{200\pi} = 0.00159 \, \mu F$

from which

$$C_2 = 2Q C = 2\left(0.00159\right) \mu F = 0.00318 \, \mu F$$

$$C_3 = C/2Q = 0.5\left(0.00159\right) \mu F = 0.000795 \, \mu F$$

The multiple feedback circuit of Figure 82.7b is a minimum component count, two-pole (or one-pole pair), bandpass filter circuit with user definable gain. It cannot generate stopband zeros and therefore is limited in its use to monotonic roll-off transfer functions. Phase shift of its amplifier reduces the $Q$ of the section and shifts the $f_p$. It is recommended to use an amplifier whose open loop gain at $f_p$ is $> 100Q^2 H_p$.

The design equations for the multiple feedback circuit of Figure 82.4b are

$$H(s) = \frac{s}{R C_1 s^2 + \frac{s}{R C_2} + \frac{1}{C_1} + \frac{1}{C_2} + \frac{R + R_2}{R R_2 R_3 C_1 C_2}} = \frac{s Q \omega_p H_p}{s^2 + \frac{s \omega_p}{Q} + \frac{\omega_p^2}{Q}}$$

(82.36)

when $s = j\omega_p$ the gain $H_p$ is

$$H_p = \frac{R C_2}{R \left(C_1 + C_2\right)}$$

(82.37)

From Equation 82.36 and 82.37 for a required set of $\omega_p$, $Q$, and $H_p$:

$$R_1 = \frac{Q}{C_1 H_p \omega_p}, \quad R_2 = \frac{Q}{\omega_p^2 \left(Q^2 \left(C_1 + C_2\right) - H_p C_1\right)}, \quad R_3 = \frac{R H_p \left(C_1 + C_2\right)}{C_2}$$

(82.38)

For $R_2$ to be realizable,

$$Q^2 \left(C_1 + C_2\right) \geq H_p C_1$$

(82.39)

The design sequence for a multiple feedback bandpass filter is as follows

Select $C_1$ and $C_2$ to satisfy Equation 82.39 for the $H_p$ and $Q$ required. Compute $R_1$, $R_2$, and $R_3$. Scale $R_1$, $R_2$, $R_3$, $C_1$, and $C_2$ as required to meet desired impedance levels.

Note that it is common to use $C_1 = C_2 = C$ for applications where $H_p = 1$ and $Q > 0.707$.
The *state variable* circuit of Figure 82.7c is the most widely used active filter circuit. It is the basic building block of programmable active filters and of switched capacitor designs. While it uses three or four amplifiers and numerous other circuit elements to realize a two-pole filter section, it has many desirable features. From a single input it provides low-pass \(V_L\), high-pass \(V_H\), and bandpass \(V_B\) outputs and by summation into an additional amplifier \(A_4\) (or the input stage of the next section) a band reject \(V_R\) or stop band zero can be created. Its two integrator resistors connect to the virtual ground of their amplifiers \(A_2, A_3\) and therefore have no signal swing on them. Therefore, programming resistors can be switched to these summing junctions using electronic switches. The sensitivity of the circuit to the gain and phase performance of its amplifiers is more than an order of magnitude less than single amplifier designs. The open-loop gain at \(f_p\) does not have to be multiplied by either the desired \(Q\) or the gain at dc or \(f_0\). Second-order sections with \(Q\) up to 100 and \(f_p\) up to 1 MHz can be built with this circuit.

There are several possible variations of this circuit that improve its performance at particular outputs. The input can be brought into several places to create or eliminate phase of inversions; the damping feedback can be implemented in several ways other than the \(R_Qa\) and \(R_{Qb}\) that are shown in Figure 82.7c and the \(f_p\) and \(Q\) of the section can be or adjusted independently from one another. DC offset adjustment components can be added to allow the offset at any one output to be trimmed to zero.

For simplicity of presentation, Figure 82.7c makes several of the resistors equal and identifies others with subscripts that relate to their function in the circuit. Specifically, the feedback amplifier \(A_1\), that generates the \(V_H\) output has equal feedback and input resistor from the \(V_L\) feedback signal to create unity gains from that input. Similarly, the “zero summing” amplifier, \(A_4\) has equal resistors for its feedback and input from \(V_L\) to make the dc gain at the \(V_R\) output the same as that at \(V_L\). More general configurations with all elements included in the equation of the transfer function are available in numerous reference texts including Reference 20.

The *state variable* circuit, as configured in Figure 82.7c, has four outputs. Their transfer functions are

\[
V_L(s) = -\frac{R}{R\left(RC\right)^2} \left( \frac{1}{D(s)} \right)
\]

(82.40a)

\[
V_H(s) = \frac{R}{R} \left( \frac{s}{R\left(RC\right)} \right) \left( \frac{s}{D(s)} \right)
\]

(82.40b)

\[
V_{ii}(s) = -\frac{R}{R} \left( \frac{s^2}{D(s)} \right)
\]

(82.40c)

\[
V_i(s) = \frac{R}{R\left(RC\right)^2} \left( \frac{\left(\frac{R}{R}\right)s^2+1}{D(s)} \right)
\]

(82.40d)

where

\[
D(s) = s^2 + \frac{a}{R\left(RC\right)} s + \frac{1}{(R\left(RC\right))^2} = s^2 + \frac{\omega_p}{Q} s + \omega_p^2 \quad a = \frac{R_{Qb}}{R_{Qa} + R_{Qb}} \left( 2 + \frac{R}{R} \right)
\]

(82.41)
Note that the dc gain at the low-pass output is

\[ V_L(0) = -\frac{R}{R_i} \tag{82.42} \]

from which obtains

\[ \omega_p = \frac{1}{R_i C} \quad \text{and} \quad \frac{1}{Q} = \frac{R_{ab}}{R_{ab} + R_{Qb}} \left( 2 + \frac{R}{R_i} \right) \tag{82.42} \]

The design sequence for the state variable filter of Figure 82.7c is

Select the values of \( R_f \) and \( C \) to set the frequency \( \omega_p \), the values of \( R_i \) for the desired dc gain and \( R_{Qa} \) and \( R_{Qb} \) for the desired \( Q \) and dc gain.

82.5 Discrete-Time Filters

A digital filter is a circuit or a computer program that computes a discrete output sequence from a discrete input sequence. Digital filters belong to the class of discrete-time LTI (linear time invariant) systems, which are characterized by the properties of causality, recursibility, and stability, and may be characterized in the time domain by their impulse response and in the transform domain by their transfer function. The most general case of a discrete-time LTI system with the input sequence denoted by \( x(kT) \) and the resulting output sequence \( y(kT) \) can be described by a set of linear difference equations with constant coefficients.

\[ y(kT) = \sum_{\mu=0}^{N} b_{\mu} x(kT - \mu T) - \sum_{\mu=1}^{N} a_{\mu} y(kT - \mu T) \tag{82.43} \]

where \( a_0 = 1 \). An equivalent relation between the input and output variables can be given through the convolution sum in terms of the impulse response sequence \( h(kT) \):

\[ y(kT) = \sum_{\mu=0}^{N} h(kT) x(kT - \mu T) \tag{82.44} \]

The corresponding transfer function is given by

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{\mu=0}^{N} b_{\mu} z^{-\mu}}{1 + \sum_{\mu=1}^{N} a_{\mu} z^{-\mu}} \quad \Leftrightarrow \quad H(z) = b_0 \prod_{\mu=1}^{N} \left( z - z_{\mu} \right) \tag{82.45} \]

where \( H(z) \) is the z-transform of the impulse response \( h(kT) \) and \( X(z) \), \( Y(z) \) are the z-transform of the input signal \( x(kT) \) and the output or the filtered signal \( y(kT) \). As can be seen from Equation 82.44, if for at least one \( \mu \), \( a_{\mu} \neq 0 \), the corresponding system is recursive; its impulse response is of infinite duration — infinite impulse response (IIR) filter. If \( a_{\mu} = 0 \), the corresponding system is nonrecursive — finite
impulse response (FIR) filter; its impulse response is of finite duration and the transfer function $H(z)$ is a polynomial in $z^{-1}$. The zeros of the polynomial $X(z)$ given by $z = z_i$ are called the poles of $H(z)$ and are commonly referred to as the natural frequencies of the filter. The condition for the stability of the filter is expressed by the constraint that all the poles of $H(z)$ should lie inside the unit circle, that is $|z_i| < 1$. The zeros of $Y(z)$ given by $z = z_n$ which are equivalent to the zeros of $H(z)$ are called the transmission zeros of the filter. Clearly, at these frequencies the output of the filter is zero for any finite input.

On the unit circle, the transfer function frequency $H(z)$ reduces to the frequency response function $H(e^{j\omega T})$, the discrete-time Fourier transform of $h(kT)$, which in general is complex and can be expressed in terms of magnitude and phase

$$H(e^{j\omega T}) = |H(e^{j\omega T})|e^{j\phi(\omega)}$$

(82.46)

The gain function of the filter is given as

$$\alpha(\Omega) = 20 \log_{10}|H(e^{j\omega T})|$$

(82.47)

It is also common practice to call the negative of the gain function the attenuation. Note that the attenuation is a positive number when the magnitude response is less than 1.

Figure 82.9 gives a block diagram realizing the difference equation of the filter, which is commonly referred to as the direct-form I realization. Notice that the element values for the multipliers are obtained directly from the numerator and denominator coefficients of the transfer function. By rearranging the structure in regard to the number of delays, one can obtain the canonic structure called direct-form II shown in Figure 82.10, which requires the minimum number of delays.

Physically, the input numbers are samples of a continuous signal and real-time digital filtering involves the computation of the iteration of Equation 82.43 for each incoming new input sample. Design of a filter consists of determining the constants $a_k$ and $b_k$ that satisfies a given filtering requirement. If the filtering is performed in real time, then the right side of Equation 82.46 must be computed in less than the sampling interval $T$. 

---

**FIGURE 82.9** The difference equation of a digital filter can be realized by a direct-form I implementation that uses separate delay paths for the $X$ and $Y$ summations.
82.6 Digital Filter Design Process

The digital filter design procedure consists of the following basic steps:

1. Determine the desired response. The desired response is normally specified in the frequency domain in terms of the desired magnitude response and/or the desired phase response.
2. Select a class of filters (e.g., linear-phase FIR filters or IIR filters) to approximate the desired response.
3. Select the best member in the filter class.
4. Implement the best filter using a general-purpose computer, a DSP, or a custom hardware chip.
5. Analyze the filter performance to determine whether the filter satisfies all the given criteria.

82.7 FIR Filter Design

In many digital signal-processing applications, FIR filters are generally preferred over their IIR counterparts, because they offer a number of advantages compared with their IIR equivalents. Some of the good properties of FIR filters are a direct consequence of their nonrecursive structure. First, FIR filters are inherently stable and free of limit cycle oscillations under finite-word length conditions. In addition, they exhibit a very low sensitivity to variations in the filter coefficients. Second, the design of FIR filters with exactly linear phase (constant group delay) vs. frequency behavior can be accomplished easily. This property is useful in many application areas, such as speech processing, phase delay equalization, image processing, etc.

Finally, there exists a number of efficient algorithms for designing optimum FIR filters with arbitrary specifications. The main disadvantage of FIR filters over IIR filter is that FIR filter designs generally require, particularly in applications requiring narrow transition bands, considerably more computation to implement.

An FIR filter of order \( N \) is described by a difference equation of the form

\[
y(kT) = \sum_{\mu=0}^{N} b_{\mu} x(kT - \mu T) \tag{82.48}
\]
and the corresponding transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{\mu=0}^{N} b_{\mu} z^{-\mu} \tag{82.49}$$

The objective of FIR filter design is to determine $N \pm 1$ coefficients given by

$$h(0), h(1), \ldots, h(N) \tag{82.50}$$

so that the transfer function $H(e^{j\omega T})$ approximates a desired frequency characteristic. Note that because Equation 82.47 is also in the form of a convolution summation, the impulse response of an FIR filter is given by

$$h(kT) = \begin{cases} b_{\mu}, & k = 0, 1, \ldots, N \\ 0, & \text{otherwise} \end{cases} \tag{82.51}$$

Two equivalent structures for FIR filters are given in Figure 82.11. The accuracy of an FIR approximation is described by the following parameters:

- $\delta_p$ passband ripple
- $\delta_s$ stopband attenuation
- $\Delta\omega$ transition bandwidth

These quantities are depicted in Figure 82.12 for a prototype low-pass filter. $\delta_p$ and $\delta_s$ characterize the permissible errors in the passband and in stopband, respectively. Usually, the passband ripple and stopband attenuation are given in decibels, in which case their values are related to the parameters $\delta_p$ and $\delta_s$ by

$$\text{Passband ripple (dB): } A_p = -20 \log_{10}(1 - \delta_p) \tag{82.52}$$
Note that due to the symmetry and periodicity of the magnitude response of $H(e^{j\omega T})$, it is sufficient to give the filter specifications in the interval $0 \leq \omega \leq \pi$.

**Windowed FIR Filters**

Several design techniques can be employed to synthesize linear-phase FIR filters. The simplest implementation is based on windowing, which commonly begins by specifying the ideal frequency response and expanding it in a Fourier series and then truncating and smoothing the ideal impulse response by means of a window function. The truncation results in large ripples before and after the discontinuity of the ideal frequency response known as the Gibbs phenomena, which can be reduced by using a window function that tapers smoothly at both ends. Filters designed in this way possess equal passband ripple and stopband attenuation, i.e.,

$$\delta_p = \delta_s = \delta$$  \hspace{1cm} (82.54)

To illustrate this method, let us define an ideal desired frequency response that can be expanded in a Fourier series

$$H_d(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} h_d(kT) e^{-j\omega kT}$$  \hspace{1cm} (82.55)

where $h_d(kT)$ is the corresponding impulse response sequence, which can be expressed in terms of $H_d(e^{j\omega T})$ as

$$h_d(kT) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega T}) e^{j\omega kT} d\omega$$  \hspace{1cm} (82.56)

The impulse response of the desired filter is then found by weighting this ideal impulse response with a window $w(kT)$ such that
Note that for \( w(kT) \) in the above-given interval we obtain the rectangular window. Some commonly used windows are Bartlett (triangular), Hanning, Hamming, Blackmann, etc., the definitions of which can be found in Reference 15.

As an example of this design method, consider a low-pass filter with a cutoff frequency of \( \omega_c \) and a desired frequency of the form

\[
H_d(e^{j\omega_T}) = \begin{cases} 
    e^{-j\omega_T/2}, & |\omega| \leq \omega_c \\
    0, & \omega_c < |\omega| \leq \pi,
\end{cases}
\]  

Using Equation 82.56 we obtain the corresponding ideal impulse response

\[
h_d(kT) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega_T/2} e^{j\omega kT} d\omega = \frac{\sin\left[\omega \left( kT - TN/2 \right) \right]}{\pi \left( kT - TN/2 \right)}
\]

Choosing \( N = 4 \), \( \omega_c = 0.6\pi \) and a Hamming window defined by

\[
w(kT) = \begin{cases} 
    0.54 - 0.46 \cos\left(2\pi kT/N\right), & 0 \leq k \leq N \\
    0, & \text{otherwise}
\end{cases}
\]

we obtain the following impulse response coefficients:

\[
\begin{align*}
    h(0) &= -0.00748 \\
    h(1) &= 0.12044 \\
    h(2) &= -0.54729 \\
    h(3) &= 0.27614 \\
    h(4) &= -0.03722
\end{align*}
\]  

**Optimum FIR Filters**

As mentioned earlier, one of the principal advantages of FIR filters over their IIR counterparts is the availability of excellent design methods for optimizing arbitrary filter specifications. Generally, the design criterion for the optimum solution of an FIR filter design problem can be characterized as follows. The maximum error between the approximating response and the given desired response has to be minimized, i.e.,

\[
E\left(e^{j\omega_T}\right) = W_d\left(e^{j\omega_T}\right) \left| H_d\left(e^{j\omega_T}\right) - H\left(e^{j\omega_T}\right) \right|
\]  

\[\text{(82.62)}\]
where $E(e^{j\omega})$ is the weighted error function on a close range $X$ of $[0, \pi]$ and $W(e^{j\omega})$ a weighting function, which emphasizes the approximation error parameters in the design process. If the maximum absolute value of this function is less than or equal $\varepsilon$ on $X$, i.e.,

$$\varepsilon = \max_{\omega \in X} |E(e^{j\omega})|$$  \hspace{1cm} (82.63)

the desired response is guaranteed to meet the given criteria. Thus, this optimization condition implies that the best approximation must have an equiripple error function. The most frequently used method for designing optimum magnitude FIR filters is the Parks–McClellan algorithm. This method essentially reduces the filter design problem into a problem in polynomial approximation in the Chebyshev approximation sense as discussed above. The maximum error between the approximation and the desired magnitude response is minimized. It offers more control over the approximation errors in different frequency bands than is possible with the window method. Using the Parks–McClellan algorithm to design FIR filters is computationally expensive. This method, however, produces optimum FIR filters by applying time-consuming iterative techniques. A FORTRAN program for the Parks–McClellan algorithm can be found in the IEEE publication Programs for DSP in Reference 12. As an example of an equiripple filter design using the Parks–McClellan algorithm, a sixth-order low-pass filter with a passband $0 \leq \omega \leq 0.6\pi$, a stopband $0.8\pi \leq \omega \leq \pi$, and equal weighting for each band was designed by means of this program.

The resulting impulse response coefficients are

$$
\begin{align*}
  h(0) &= h(6) = -0.00596 \\
  h(1) &= h(5) = -0.18459 \\
  h(2) &= h(4) = 0.25596 \\
  h(3) &= 0.70055
\end{align*}
$$  \hspace{1cm} (82.64)

### Design of Narrowband FIR Filters

When using conventional techniques to design FIR filters with especially narrow bandwidths, the resulting filter lengths may be very high. FIR filters with long filter lengths often require lengthy design and implementation times, and are more susceptible to numerical inaccuracy. In some cases, conventional filter design techniques, such as the Parks–McClellan algorithm, may fail the design altogether. A very efficient algorithm called the interpolated finite impulse response (IFIR) filter design technique can be employed to design narrowband FIR filters. Using this technique produces narrowband filters that require far fewer coefficients than those filters designed by the direct application of the Parks–McClellan algorithm. For more information on IFIR filter design, see Reference 7.

### 82.8 IIR Filter Design

The main advantage of IIR filters over FIR filters is that IIR filters can generally approximate a filter design specification using a lower-order filter than that required by an FIR design to perform similar filtering operations. As a consequence, IIR filters execute much faster and do not require extra memory, because they execute in place. A disadvantage of IIR filters, however, is that they have a nonlinear phase response. The two most common techniques used for designing IIR filters will be discussed in this section. The first approach involves the transformation of an analog prototype filter. The second method is an optimization-based approach allowing the approximation of an arbitrary frequency response.
The transformation approach is quite popular because the approximation problem can be reduced to the design of classical analog filters, the theory of which is well established, and many closed-form design methods exist. Note that this is not true for FIR filters, for which the approximation problems are of an entirely different nature. The derivation of a transfer function for a desired filter specification requires the following three basic steps:

1. Given a set of specifications for a digital filter, the first step is to map the specifications into those for an equivalent analog filter.
2. The next step involves the derivation of a corresponding analog transfer function for the analog prototype.
3. The final step is to translate the transfer function of the analog prototype into a corresponding digital filter transfer function.

Once the corresponding analog transfer function for the analog prototype is derived, it must be transformed using a transformation that maps $H_a(s)$ into $H(z)$. The simplest and most appropriate choice for $s$ is the well-known bilinear transform of the $z$-variable

$$s = \frac{2\left(1-z^{-1}\right)}{T_d\left(1+z^{-1}\right)} \iff z = \frac{1+(T_d/2)s}{1-(T_d/2)s}$$  \hspace{1cm} (82.65)$$

which maps a stable analog filter in the $s$-plane into a stable digital filter in the $z$-plane. Substituting $s$ with the right-hand side of Equation 82.63 in $H_a(s)$ results in

$$H(z) = H_a\left(\frac{2\left(1-z^{-1}\right)}{T_d\left(1+z^{-1}\right)}\right) \Rightarrow H\left(e^{j\omega T}\right) = H\left(\frac{2j}{T_d} \tan\left(\frac{\omega T}{2}\right)\right)$$  \hspace{1cm} (82.66)$$

As it can be seen from Equation 82.66, the analog frequency domain (imaginary axis) maps onto the digital frequency domain (unit circle) nonlinearly. This phenomena is called frequency warping and must be compensated in a practical implementation. For low frequencies $\Omega$ and $\omega$ are approximately equal. We obtain the following relation between the analog frequency $\Omega$ and the digital frequency $\omega$

$$\Omega = \frac{2}{T_d} \tan\left(\frac{\omega T}{2}\right)$$  \hspace{1cm} (82.67)$$

$$\omega = \frac{2}{T} \arctan\left(\frac{\Omega T_d}{2}\right)$$  \hspace{1cm} (82.68)$$

The overall bilinear transformation procedure is as follows:

1. Convert the critical digital frequencies (e.g., $\omega_p$ and $\omega_s$ for low-pass filters) to the corresponding analog frequencies in the $s$-domain using the relationship given by Equation 82.67.
2. Derive the appropriate continuous prototype transfer function $H_a(s)$ that has the properties of the digital filter at the critical frequencies.
3. Apply the bilinear transform to $H_a(s)$ to obtain $H(z)$ which is the required digital filter transfer function.

To illustrate the three-step IIR design procedure using the bilinear transform, consider the design of a second-order Butterworth low-pass filter with a cutoff frequency of $\omega_c = 0.5\pi$. The sampling rate of the digital filter is to be $f_s = 10$ Hz, giving $T = 0.1$ s. First, we map the cutoff frequency to the analog frequency...
The poles of the analog Butterworth filter transfer function $H_a(s)$ are found using Equation 82.11. As explained earlier, these poles lie equally spaced in the $s$-plane on a circle of radius $\Omega_c$.

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}\Omega_cs + \Omega_c^2}$$  \hspace{1cm} (82.70)

Application of the bilinear transformation

$$s = \frac{2(1-z^{-1})}{0.1(1+z^{-1})}$$  \hspace{1cm} (82.71)

gives the digital transfer function

$$H(z) = \frac{0.00002 + 0.00004z^{-1} + 0.00002z^{-2}}{1-1.98754z^{-1} + 0.98762z^{-2}}$$  \hspace{1cm} (82.72)

The above computations were carried out using Reference 9, which greatly automates the design procedure.

### Design of Arbitrary IIR Filters

The IIR filter design approach discussed in the previous section is primarily suitable for frequency-selective filters based on closed-form formulas. In general, however, if a design other than standard low-pass, high-pass, bandpass, and stopband is required, or if the frequency responses of arbitrary specifications are to be matched, in such cases it is often necessary to employ algorithmic methods implemented on computers. In fact, for nonstandard response characteristics, algorithmic procedures may be the only possible design approach. Depending on the error criterion used, the algorithmic approach attempts to minimize the approximation error between the desired frequency response $H_d(e^{j\omega T})$ and $H(e^{j\omega T})$ or between the time-domain response $h_d(kT)$ and $h(kT)$. Computer software is available for conveniently implementing IIR filters approximating arbitrary frequency response functions [8,9].

### Cascade-Form IIR Filter Structures

Recall that theoretically there exist an infinite number of structures to implement a digital filter. Filters realized using the structure defined by Equation 82.44 directly are referred to as direct-form IIR filters. The direct-form structure, however, is not employed in practice except when the filter order $N \leq 2$, because they are known to be sensitive to errors introduced by coefficient quantization and by finite-arithmetic conditions. Additionally, they produce large round-off noise, particularly for poles close to the unit circle.

Two less-sensitive structures can be obtained by partial fraction expansion or by factoring the right-hand side of Equation 82.46 in terms of real rational functions of order 1 and 2. The first method leads to parallel connections and the second one to cascade connections of corresponding lower-order sections, which are used as building blocks to realize higher-order transfer functions. In practice, the cascade form is by far the preferred structure, since it gives the freedom to choose the pairing of numerators and denominators and the ordering of the resulting structure. Figure 82.13 shows a cascade-form implementation, whose overall transfer function is given by
where the transfer function of the kth building block is

\[ H_k(z) = \frac{b_{2k} z^{-1} + b_{1k} z^{-2}}{1 + a_n z^{-2} + a_{2n} z^{-2}} \]  \hspace{1cm} (82.74)

Note this form is achieved by factoring Equation 82.45 into second-order sections.

There are, of course, many other realization possibilities for IIR filters, such as state-space structures [9], lattice structures [10], and wave structures. The last is introduced in the next section.

### 82.9 Wave Digital Filters

It was shown earlier that for recursive digital filters the approximation problem can be reduced to classical design problems by making use of the bilinear transform. For wave digital filters (WDFs) this is carried one step farther in that the structures are obtained directly from classical circuits. Thus, to every WDF there corresponds an LCR reference filter from which it is derived. This relationship accounts for their excellent properties concerning coefficient sensitivity, dynamic range, and all aspects of stability under finite-arithmetic conditions. The synthesis of WDFs is based on the wave network characterization; therefore, the resulting structures are referred to as wave digital filters. To illustrate the basic idea behind the theory of WDFs, consider an inductor \( L \), which is electrically described by \( V(s) = sL I(s) \). In the next step we define wave variables \( A_i(s) \) and \( B_i(s) \) as

\[
\begin{align*}
A_i(s) & = V(s) + RI(s) \\
B_i(s) & = V(s) - RI(s)
\end{align*}
\]  \hspace{1cm} (82.75)

where \( R \) is called the port resistance. Substituting \( V(s) = sL I(s) \) in the above relation and replacing \( s \) in \( A_i(s) \) and \( B_i(s) \) with the bilinear transform given by Equation 82.65, we obtain

\[
B(z) = \frac{1-z^{-1}}{1-z^{-1}} \frac{1-z^{-1}}{1+L} R A(z) \]  \hspace{1cm} (82.76)

Letting \( R = L \), the above relation reduces to

\[
B(z) = -z^{-1} A(z) \]  \hspace{1cm} (82.77)

Thus an inductor translates into a unit delay in cascade with an inverter in the digital domain. Similarly, it is easily verified that a capacitance can be simulated by a unit delay and a resistor by a digital sink. Figure 82.14 shows the digital realizations of impedances and other useful one-port circuit elements.
To establish an equivalence with classical circuits fully, the interconnections are also simulated by so-called wave adaptors. The most important of these interconnections are series and parallel connections, which are simulated by series and parallel adaptors, respectively. For most filters of interest, only two- and three-port adaptors are employed. For a complete design example consider Figure 82.15.

FIGURE 82.14 Digital filter implementations use functional equivalents to one port linear filter elements.

FIGURE 82.15 Digital wave filters establish equivalence with classical filter circuits by use of wave adapter substitutions: (A) $LC$ reference low-pass; (B) identification of wire interconnections; (C) corresponding wave digital filter.
For a given LC filter, one can readily derive a corresponding WDF by using the following procedure. First, the various interconnections in the LC filter are identified as shown in Figure 82.15. In the next step, the electrical elements in the LC filter are replaced by its digital realization using Figure 82.15. Finally, the interconnections are substituted using adaptors. Further discussions and numerical examples dealing with WDFs can be found in Reference 3, 13, and 14.

82.10 Anti-Aliasing and Smoothing Filters

In this section two practical application areas of filters in the analog conditioning stage of a data acquisition system are discussed. A block diagram of a typical data acquisition system is shown in Figure 82.16, consisting of an antialiasing filter before the analog-to-digital converter (ADC) and a smoothing filter after the digital-to-analog converter (DAC).

For a complete discrete reconstruction of a time-continuous, band-limited input signal having the spectrum $0 \leq f \leq f_{\text{max}}$, the sampling frequency must be, according to the well-known Shannon’s sampling theorem, at least twice the highest frequency in the signal. In our case, in order to be able to represent frequencies up to $f_{\text{max}}$, the sampling frequency $f_s = 1/T > 2f_{\text{max}}$. The necessary band limiting to $f \leq f_{\text{max}}$ of the input time-continuous signal is performed by a low-pass filter, which suppresses higher spectral components greater than $f_{\text{max}}$. Violation of this theorem results in alias frequencies. As a result, frequency components above $f_s/2$, the so-called Nyquist frequency, appear as frequency components below $f_s/2$.

Aliasing is commonly addressed by using antialiasing filters to attenuate the frequency components at and above the Nyquist frequency to a level below the dynamic range of an ADC before the signal is digitized. Ideally, a low-pass filter with a response defined by

$$H(j\Omega) = \begin{cases} 1, & \Omega \leq \pi/T \\ 0, & \Omega \geq \pi/T \end{cases}$$  (82.78)

is desired to accomplish this task. In practice, a variety of techniques based on the principles of continuous-time analog low-pass filter design can be employed to approximate this “brick-wall” type of characteristic. Anti-aliasing filters typically exhibit attenuation slopes in the range from 45 to 120 dB/octave and stopband rejection from 75 to 100 dB. Among the types of filters more commonly used for antialiasing purposes are the Cauer elliptic, Bessel, and Butterworth. The optimum type of filter depends on which kinds of imperfections, e.g., gain error, phase nonlinearity, passband and stopband ripple, etc., are most likely to be tolerated in a particular application. For example, Butterworth filters exhibit very flat frequency response in the passband, while Chebyshev filters provide steeper attenuation at the expense of some passband ripple. The Bessel filter provides a linear phase response over the entire passband but less attenuation in the stopband. The Cauer elliptic filter, with its extremely sharp roll-off, is especially useful as an anti-aliasing filter for multichannel digitizing data acquisition systems. However, the large-phase nonlinearity makes it more appropriate for applications involving analysis of the frequency content of signals as opposed to phase content or waveform shape.

Many considerations discussed above also apply to smoothing filters. Due to the sampling process, the frequency response after the digital-to-analog conversion becomes periodic with a period equal to the sampling frequency. The quantization steps that are created in the DAC reconstruction of the output waveform and are harmonically related to the sampling frequency must be suppressed through a low-
pass filter having the frequency response of Equation 82.78 also referred to as a smoothing or reconstruction filter. While an antialiasing filter on the input avoids unwanted errors that would result from undersampling the input, a smoothing filter at the output reconstructs a continuous-time output from the discrete-time signal applied to its input.

Consideration must be given to how much antialiasing protection is needed for a given application. It is generally desirable to reduce all aliasable frequency components (at frequencies greater than half of the sampling frequency) to less than the LSB of the ADC being used. If it is possible that the aliasable input can have an amplitude as large as the full input signal range of the ADC, then it is necessary to attenuate it by the full $2^N$ range of the converter. Since each bit of an ADC represents a factor of 2 from the ones adjacent to it, and $20 \log(2) = 6$ dB, the minimum attenuation required to reduce a full-scale input to less than a LSB is

$$
\alpha < -20N(\text{dB})
$$

(82.79)

where $N$ is the number of bits of the ADC.

The amount of attenuation required can be reduced considerably if there is knowledge of the input frequency spectrum. For example, some sensors, for reasons of their electrical or mechanical frequency response, might not be able to produce a full-scale signal at or above the Nyquist frequency of the system and therefore “full-scale” protection is not required. In many applications, even for 16-bit converters that, in the worst case, would require 96 dB of antialias protection, 50 to 60 dB is adequate.

Additional considerations in antialias protection of the system are the noise and distortion that are introduced by the filter that is supposed to be eliminating aliasable inputs. It is possible to have a perfectly clean input signal which, when it is passed through a prefilter, gains noise and harmonic distortion components in the frequency range and of sufficient amplitude to be within a few LSBs of the ADC. The ADC cannot distinguish between an actual signal that is present in the input data and a noise or distortion component that is generated by the prefilter. It is necessary that both noise and distortion components in the output of the antialias filter must also be kept within an LBS of the ADC to ensure system accuracy.

### 82.11 Switched Capacitor Filters

Switched-capacitor (SC) filters, also generally referred to as analog sampled data filters, provide an alternative to conventional active-RC filters and are commonly used in the implementation of adjustable antialiasing filters. SC filters comprise switches, capacitors, and op amps. Essentially, an SC replaces the resistor in the more traditional analog filter designs. Because the impedance of the SC is a function of the switching frequency, one can vary the cutoff frequency of the SC filter by varying the frequency of the clock signal controlling the switching. The main advantage of SC filters is that they can be implemented in digital circuit process technology, since the equivalent of large resistors can be simulated by capacitors having small capacitance values.

When using SC filters, one must also be aware that they are in themselves a sampling device that requires antialias protection on the input and filtering on their outputs to remove clock feedthrough. However, since clock frequencies are typically 50 to 100 times $f_c$ of the filter, a simple first or second RC filter on their inputs and outputs will reduce aliases and noise sufficient to permit their use with 12- to 14-bit ADCs. One need also to consider that they typically have dc offset errors that are large, vary with time, temperature, and programming or clock frequency. Interested readers may refer to References 5 and 14.

### 82.12 Adaptive Filters

Adaptive filtering is employed when it is necessary to realize or simulate a system whose properties vary with time. As the input characteristics of the filter change with time, the filter coefficients are varied with...
time as a function of the filter input. Some typical applications of adaptive filtering include spectral estimation of speech, adaptive equalization, echo cancellation, and adaptive control, to name just a few. Depending on the application, the variations in the coefficients are carried out according to an optimization criterion and the adaptation is performed at a rate up to the sampling rate of the system. The self-adjustment capability of adaptive filter algorithms is very valuable when the application environment cannot be precisely described. Some of the most widely used adaptive algorithms are LMS (least-mean square), RLS (recursive least-squares), and frequency domain, also known as block algorithm. The fundamental concept of an adaptive filter is depicted in Figure 82.17.

An adaptive filter is characterized by the filter input $x(kT)$ and the desired response $d(kT)$. The error sequence $e(kT)$ formed by

$$e(kT) = \sum_{\mu=0}^{N-1} w_{\mu}(kT)x(kT - \mu T)$$  \hspace{1cm} (82.80)

and $x(kT), \ldots, x(kT - T(N-1))$ serve as inputs to an adaptive algorithm that recursively determines the coefficients $w_{0}(kT + T), \ldots, w_{N-1}(kT + T)$. A number of adaptive algorithms and structures can be found in the literature that satisfy different optimization criteria in different application areas. For more detailed developments refer to References 1, 15, and 16.

**Defining Terms**

**Antialiasing filter:** Antialiasing filters remove any frequency elements above the Nyquist frequency. They are employed before the sampling operation is conducted to prevent aliasing in the sampled version of the continuous-time signal.

**Bandpass filter:** A filter whose passband extends from a lower cutoff frequency to an upper cutoff frequency. All frequencies outside this range are stopped.

**Equiripple:** Characteristic of a frequency response function whose magnitude exhibits equal maxima and minima in the passband.

**Finite impulse response (FIR) filter:** A filter whose response to a unit impulse function is of finite length, i.e., identically zero outside a finite interval.

**High-pass filter:** A filter that passes all frequencies above its cutoff frequency and stops all frequencies below it.

**Ideal filter:** An ideal filter passes all frequencies within its passband with no attenuation and rejects all frequencies in its stopband with infinite attenuation. There are five basic types of ideal filters: low pass, high pass, bandpass, stopband, and all pass.
Infinite impulse response (IIR) filter: A filter whose response to a unit impulse function is of infinite length, i.e., nonzero for infinite number of samples.

Low-pass filter: A filter that attenuates the power of any signals with frequencies above its defined cutoff frequency.

Passband: The range of frequencies of a filter up to the cutoff frequency.

Stopband: The range of frequencies of a filter above the cutoff frequency.

Transition region: The range of frequencies of a filter between a passband and a stopband.

References

R. B. Panerai, et. al.. "Spectrum Analysis and Correlation."

Most sensors and instruments described in previous sections of this handbook can produce continuous measurements in time or sequential measurements at fixed or variable time intervals, as represented in Figure 83.1. The temporal patterns resulting from such measurements are usually referred to as signals.

Signals can either be continuous or discrete in time (Figure 83.1). The main objective of spectral analysis is to provide an estimate of the distribution of signal power at different frequencies. Spectral analysis and correlation techniques are an aid to the interpretation of signals and to the systems that generate them. These methods are now widely used for the analysis and interpretation of measurements performed in medicine, geophysics, vibration analysis, communications, and several other areas.

Although the original concept of a signal involves measurements as a function of time (Figure 83.1), this term has been generalized to include measurements along other dimensions, e.g., distance. In addition, signals can have multiple dimensions — the instantaneous velocity of an airplane can be regarded as a four-dimensional signal since it depends on time and three spatial coordinates.

With the growing availability of signal-processing computer packages and dedicated instruments, most readers will perform spectral analysis and correlation at the "touch of a button," visualizing results on a screen or as a computer plot. These "black-box" systems are useful for saving time and money, but users should be aware of the limitations of the fundamental techniques and circumstances in which inappropriate use can lead to misleading results. This chapter presents the basic concepts of spectral analysis and correlation based on the fast Fourier transform (FFT) approach. FFT algorithms allow the most efficient computer implementation of methods to perform spectral analysis and correlation and have become the most popular option. Nevertheless, other approaches, such as parametric techniques, wavelet transforms, and time-frequency analysis are also available. These will be briefly discussed and the interested reader will be directed to the pertinent literature for applications that might benefit from alternative approaches.
Fundamental Concepts

Spectral Analysis

Practical applications of spectral and correlation analysis are performed on discrete-time signals (Figure 83.1). These are obtained either from a sequence of discrete measurements or from the transformation of a continuous signal (Figure 83.1) to digital format using an analog-to-digital converter (ADC).

FIGURE 83.1 Examples of continuous and discrete-time signals. (a) Continuous recording of intracranial pressure in a head-injured patient. (b) Intracranial pressure measurements obtained at regular intervals of 50 ms. (c) Non-uniformly spaced measurements of mean intracranial pressure over a period of 30 h following surgery.
When the latter is adopted to allow computer analysis of an originally continuous signal, two main characteristics of the ADC need to be considered. The first is the number of bits available to represent each sample, as this will determine the resolution and accuracy of the sampled signal. The second important consideration is the sampling interval \( \Delta t \) (Figure 83.1). From the Nyquist theorem, the maximum value of \( \Delta t \) must be such that the sampling frequency \( f_s = 1/\Delta t \) is at least twice the highest frequency of interest in the original signal. If this rule is not followed, spectral and correlation estimations might be considerably distorted by a phenomenon called aliasing. Low-pass filtering before ADC is always recommended to limit the bandwidth of the continuous signal to allow the correct choice of \( f_s \) or \( \Delta t \). In practice, the sampling frequency is usually much higher than the minimum required by the Nyquist theorem to provide a better visual representation of the sampled data.

Let \( x_n \) represent a discrete-time signal with samples at \( n = 0, 1, 2, ..., N-1 \). The Fourier theorem states that it is possible to decompose \( x_n \) as a sum of cosine and sine waveforms of different frequencies using an appropriate combination of amplitude coefficients. Therefore,

\[
x_n = a_0 + \sum_{k=1}^{N-1} a_k \cos \left( \frac{2\pi kn}{N} \right) + \sum_{k=1}^{N-1} b_k \sin \left( \frac{2\pi kn}{N} \right)
\]

(83.1)

where \( k = 1, 2, ..., N-1 \) determines the frequency of each cosine and sine waveforms as \( f_k = k/N\Delta t \). The corresponding coefficients are calculated from

\[
a_0 = \frac{1}{N} \sum_{n=0}^{N-1} x_n
\]

(83.2a)

\[
a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n \cos \left( \frac{2\pi kn}{N} \right)
\]

(83.2b)

\[
b_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n \sin \left( \frac{2\pi kn}{N} \right)
\]

(83.2c)

Note that Equation 83.2a represents the mean value of \( x_n \) and that the argument \( 2\pi kn/N \) is the same for the direct (Equation 83.2) and inverse (Equation 83.1) discrete Fourier transforms (DFT).

From Euler’s formula, it is possible to combine the cosine and sine terms to express the DFT in exponential form:

\[
e^{j\theta} = \cos \theta + j \sin \theta
\]

(83.3)

leading to

\[
x_n = \sum_{k=0}^{N-1} c_k e^{j(2\pi kn/N)}
\]

(83.4)

with

\[
c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-j(2\pi kn/N)}
\]

(83.5)
where \( c_k \) is now a complex value related to the original cosine and sine coefficients by

\[
\begin{align*}
    c_0 &= a_0 \\
    c_k &= a_k - jb_k \quad k=1,2,...,N-1
\end{align*}
\]

A graphic representation of the \( a_k, b_k, \) or \( c_k \) coefficients for each value of \( k \) (or \( f_k \)) constitutes the frequency spectrum of \( x_n \), expressing the relative contribution of different sinusoidal frequencies to the composition of \( x_n \) (Equation 83.4). Since \( c_k \) is complex (Equation 83.6b), a more meaningful physical interpretation of the spectrum is obtained with the **amplitude** and **phase** spectra, defined as

\[
\begin{align*}
    A_k &= \left( a_k^2 + b_k^2 \right)^{1/2} = |c_k| \\
    \theta_k &= \tan^{-1}\left( -\frac{b_k}{a_k} \right)
\end{align*}
\]

Figure 83.2 shows the amplitude (or magnitude) and phase spectra for the signal in Figure 83.1a, sampled at intervals \( \Delta t = 20 \) ms. The signal was low-pass-filtered at 20 Hz before ADC. The total duration is given by \( T = N\Delta t = 5 \) s, corresponding to \( N = 250 \) samples. Before calculating the spectral coefficients, the mean value of the complete record was removed (dc term) and any linear trends were removed by fitting a straight line to the data (detrending). As will be discussed below, it is also important to apply a window to the data, to minimize the phenomenon of leakage. For \( k > N/2 \) both spectra present symmetrical values. This can be easily demonstrated from the fact that cosine (and sine\(^2\)) functions have even symmetry while sine has odd symmetry. From Equation 83.7 it follows that \( A_k \) and \( \theta_k \) have even and odd symmetry, respectively. Consequently, only half the spectral components \( (k \leq N/2) \) are required to give a complete description of \( x_n \) in the frequency domain.

The amplitude spectra indicates the combined amplitude of the cosine and sine terms to reconstruct \( x_n \); the phase spectra reflects the relative phase differences (or time delays) between the sinusoidal waveforms to generate the temporal pattern of \( x_n \). The amplitude spectra also reflects the signal power at different frequencies. For simplicity, the power spectrum can be defined as

\[
P_k = A_k^2 = |c_k|^2
\]

Direct implementation of Equation 83.8, however, leads to spectral power estimates which are biased and inconsistent. More appropriate procedures for estimating the **power spectrum** (or **power density spectrum**) will be discussed later.

Parseval’s theorem\(^5\) demonstrates that the total signal energy can be computed either in time or frequency domain:

\[
\frac{1}{N} \sum_{n=0}^{N-1} x_n^2 = \sum_{k=0}^{N-1} P_k
\]

If \( x_n \) has zero mean, the left-hand side of Equation 83.9 is the biased estimator of signal variance.\(^6\) Although most applications of spectral analysis concentrate on the characteristics of the amplitude or power spectra, it is important to bear in mind that the phase spectrum is also responsible for the temporal pattern of \( x_n \). As an example, both the Dirac impulse function and white noise have a flat, constant amplitude (or
Interpretation of the amplitude and phase spectra of both theoretical functions and sampled data is facilitated by taking into account several properties of the DFT (Equations 83.4 and 83.5), namely, symmetry, linearity, shifting, duality, and convolution. To these, a very important property of Equations 83.1 and 83.4 must be added. Since cosine and sine functions are periodic, and exist for \(-\infty < t < \infty\), Equations 83.1 and 83.4 will reconstruct \(x_n\) not only in the interval of interest \((0 \leq t \leq T)\) but also at all other multiple intervals \(pT \leq t \leq (p + 1)T (p = 0, \pm 1, \pm 2, \ldots)\). As a consequence, spectral estimations obtained with the DFT inherently assume that \(x_n\) is periodic with period \(T = N/\Delta t\). As discussed in the following sections, this property needs to be taken into account when performing spectral analysis with the DFT and FFT.

**Correlation Analysis**

The basic concept of the correlation coefficient, as a measure of the strength of linear relationship between two variables, can be extended to signal analysis with the definition of the cross-correlation function (CCF) as:

![FIGURE 83.2](image)

Amplitude and phase spectra of the intracranial pressure signal represented in Figure 83.1a after analog-to-digital conversion with a sampling interval of 20 ms. The main peak in the amplitude spectrum corresponds to the frequency of the cardiac cycle in Figure 83.1a. Wraparound of the phase spectrum is apparent in the third and 13th harmonics (arrows). Both spectra have been plotted to 10 Hz only.
where $x_n$ and $y_n$ are zero-mean, discrete-time signals defined in the interval $n = 0, 1, 2, ..., N - 1$. For each value of $p$, the cross correlation is computed by shifting $y_n$ by $p\Delta t$ and calculating the average product in Equation 83.10. If $x_n$ and $y_n$ are unrelated, the sum of positive and negative products will tend to zero. Conversely, if $y_n$ tends to follow $x_n$ but with a time delay $D$, $r_{xy}(p)$ will show a peak at $p = D/\Delta t$. This property of the CCF is illustrated in Figure 83.3. As noted by Bergland, cross correlation can be viewed as "one signal searching to find itself in another signal."

For $y_n = x_n$, $r_{xy}(p)$ becomes the autocorrelation function (ACF):

$$r_{xy}(p) = \frac{1}{N} \sum_{n=0}^{N-1} x_n y_{n-p}$$

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FIGURE 83.3 CCF between changes in arterial CO$_2$ and blood flow to the brain. Arterial CO$_2$ was estimated from end-tidal measurements and cerebral blood flow with Doppler ultrasound in the middle cerebral artery. (a) CCF and original signals (inserts). The cross-correlation value of approximately 1.0, observed at time delays near zero, reflects the similar temporal patterns between the two measurements. The negative cross correlations are obtained when either signal is shifted by approximately the duration of the plateau phase, which lasts 2 min. (b) Enlarging the scale around delay $= 0$ shows that the peak cross correlation occurs at 10 s, reflecting the time it takes for the flow to respond to the CO$_2$ change (Data kindly provided by Dr. Joanne Dumville, Mr. A. Ross Naylor, and Prof. David H. Evans, University of Leicester, U.K.)

\[
r_{xy}(p) = \frac{1}{N} \sum_{n=0}^{N-1} x_n y_{n-p} \quad p = 0, \pm 1, \pm 2, \ldots
\]
and it is intuitive that the maximum value of \( r_{xx}(p) \) occurs for \( p = 0 \) with

\[
r_{xx}(0) = \frac{1}{N} \sum_{n=0}^{N-1} x_n^2
\]  

(83.12)

which represents the signal variance or total energy. Therefore, for signals with unit standard deviation, the autocorrelation peak is equal to 1.

The Wiener–Khintchine theorem\(^9\) demonstrates that the autocorrelation function and the power spectrum constitute a Fourier transform pair, that is,

\[
S_k = \sum_{p=0}^{N-1} r_{xx}(p)e^{-j2\pi kp/N}
\]  

(83.13)

where \( S_k \) is usually called the autospectra of \( x_n \).\(^6\) Equation 83.13 indicates that it is possible to estimate the power spectra from a previous estimate of the autocorrelation function. As a transform pair, the autocorrelation function can also be derived from the autospectra by substituting \( S_k \) for \( c_k \) in Equation 83.4.

From Equation 83.11 it is clear that \( r_{xx}(p) \) has even symmetry, that is, \( r_{xx}(+p) = r_{xx}(-p) \). This property is apparent in Figure 83.4, which show the estimated autocorrelation function for the signal in Figure 83.1a. Another characteristic of ACF, which can be visualized in Figure 83.4, is the occurrence of secondary peaks reflecting the presence of an oscillatory component in \( x_n \) (Figure 83.1a).

**Fast Fourier Transform**

The FFT is not a single algorithm but rather a large family of algorithms which can increase the computational efficiency of the DFT. The main ideas behind the formulation of FFT algorithms are discussed below. A detailed description of the different algorithms that have been proposed is beyond the scope of this introduction; this can be found in References 5 through 7 and 10 through 14.

For both software and hardware implementations of Equations 83.4 and 83.5, the computational efficiency is usually expressed by the number of complex multiplications and additions required or, simply, by the *number of operations*.\(^{10}\) Straight implementation of either Equation 83.4 or 83.5 leads to \( N^2 \) operations. Typically, FFT algorithms can reduce this number to \( N \log_2 N \). For \( N = 1024 \) the FFT algorithm is 100 times faster than the direct implementation of Equation 83.4 or 83.5.
The essence of all FFT algorithms is the periodicity and symmetry of the exponential term in Equations 83.4 and 83.5, and the possibility of breaking down a transform into a sum of smaller transforms for subsets of data. Since \( n \) and \( k \) are both integers, the exponential term is periodic with period \( N \).

This is commonly represented by

\[
W_N = e^{-j(2\pi/n_N)}
\]  

(83.14)

and Equation 83.5 can be written as

\[
c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n W_N^{kn} \quad k = 0, 1, 2, \ldots, N-1
\]  

(83.15)

In many applications the terms \( W_N^{kn} \) are called twiddle factors. Assuming \( N = 8 \), calculation of the DFT with Equation 83.15 will require 64 values of \( W_N^{kn} \). Apart from the minus sign, a simple calculation can show that there are only four different values of this coefficient, respectively: 1, \( j \), \( (1 + j)/\sqrt{2} \), and \( (1 - j)/\sqrt{2} \). Consequently, only these four complex factors need to be computed, representing a significant savings in number of operations.

Most FFT algorithms are based on the principle of decimation-in-time, involving the decomposition of the original time (or frequency) sequence into smaller subsequences. To understand how this decomposition can reduce the number of operations, assume that \( N \) is even. In this case it is possible to show that Equation 83.15 can be written as:

\[
c_k = \frac{1}{N} \left( \sum_{r=0}^{(N/2)-1} x_e^r \cdot W_{N/2}^{kr} + \frac{1}{N} \sum_{r=0}^{(N/2)-1} x_o^r \cdot W_{N/2}^{kr} \right)
\]  

(83.16)

where \( x_e^r \) and \( x_o^r \) represent the even- and odd-order samples of \( x_n \), respectively. Comparing Equations 83.15 and 83.16, it is clear that the latter represents two DFTs with dimension \( N/2 \), involving \( 2(N/2)^2 \) operations rather than the \( N^2 \) operations required by Equation 83.15. This process of decimation-in-time can be carried out further to improve computational performance. In the general case, \( N \) can be decomposed into \( q \) factors:

\[
N = \prod_{i=1}^{q} r_i = r_2 \ldots r_q
\]  

(83.17)

The number of operations required is then:

\[
\text{number of operations} = N \sum_{i=1}^{q} r_i
\]  

(83.18)

In the original algorithm of Cooley and Tukey, \( r_1 = 2 \) and \( N = 2^q \). In this case the theoretical number of operations required would be \( 2Nq = 2N \log_2 N \). As pointed out in Reference 6, further improvements in efficiency are possible because of the symmetry of the twiddle factors. The efficiency gain of most FFT algorithms using radix-2, i.e., \( N = 2^q \) is

\[
\text{efficiency gain} = \frac{N^2}{N \log_2 N} = \frac{N}{\log_2 N}
\]  

(83.19)
For \( N = 1024, q = 10 \) and the efficiency gain is approximately 100. Specific applications might benefit from other decompositions of the original sequence. Cases of particular interest are radix-4 and radix-8 FFTs.\(^{14}\) However, as shown by Rabiner and Gold,\(^{11}\) (p. 585), it is not possible to generalize the superiority of radix-8 over radix-4 algorithms.

In general, most FFT algorithms accept complex \( x_n \) sequences in Equation 83.5. By limiting \( x_n \) to the most common situation of real-valued signals, it is possible to obtain more efficient algorithms as demonstrated by Sorensen et al.\(^{15}\) Uniyal\(^{16}\) performed a comparison of different algorithms for real-valued sequences showing that performance is architecture dependent. For machines with a powerful floating point processor, the best results were obtained with Brunn's algorithm.\(^{17}\)

The application of FFT algorithms for spectral and correlation analysis is discussed in the following sections.

**FFT Spectral Analysis**

For some deterministic signals, \( x_n \) can be expressed by a mathematical function and the amplitude and phase spectra can be calculated as an exact solution of Equations 83.5 and 83.7. The same is true for the power spectra (Equations 83.8 and 83.13). Examples of this exercise can be found in many textbooks.\(^{1,2,4,5,7}\)

In most practical applications, there is a need to perform spectral analysis of experimental measurements, corresponding to signals which, in general, cannot be described by simple mathematical functions. In this case the spectra has to be estimated by a numerical solution of Equations 83.5 through 83.8, which can be efficiently implemented on a digital computer with an FFT algorithm. For estimation of the power spectrum, this approach is often classified as nonparametric, as opposed to other alternatives which are based on parametric modeling of the data such as autoregressive methods.\(^{18}\) Considerable distortions can result from applications of the FFT unless attention is paid to the following characteristics and properties of the measured signal and the DFT/FFT.

**Limited Observation of Signal in Time**

Limited observation of a signal \( x_n \) in time can be seen as the multiplication of the original signal \( x_n \) by a rectangular window of duration \( T = N\Delta t \) as exemplified for a single sinusoid in Figure 83.5. The DFT assumes that \( x_n \) is periodic, with period \( T \); as mentioned previously. Instead of a single harmonic at the frequency of the original sinusoid, the power spectrum estimated with the FFT will have power at other harmonics as indicated by the spectrum in Figure 83.5c. The spectral power, which should have been concentrated on a single harmonic (Figure 83.5c, dashed line), has "leaked" to neighboring harmonics and for this reason this phenomenon is usually called leakage. The morphology of the distorted spectrum of Figure 83.5c can be explained by the fact that the Fourier transform of a rectangular window function (Figure 83.5b) is given by a \( \text{sinc} \) function \( (\sin x / x) \) which presents decreasing side lobes.\(^{1,2}\) Multiplication in time corresponds to the convolution operation in the frequency domain.\(^{1,2}\) In the general case of signals comprising several harmonics, the \( \text{sinc} \) functions will superimpose and the resulting spectrum is then a distorted version of the "true" spectrum. As the individual \( \text{sinc} \) functions superimpose to produce the complete spectrum, a \textit{picket-fence} effect is also generated.\(^{8}\) This means that spectral leakage not only adds spurious power to neighboring harmonics but also restricts the frequency resolution of the main spectral peaks. The effects of spectral leakage can be reduced by (1) increasing the period of observation and (2) multiplying the original signal \( x_n \) by a window function with a smooth transition as represented by the dashed line window in Figure 83.5b. The Fourier transform of a window function with tapered ends has smaller side lobes, thus reducing the undesirable effects leakage. A large number of tapering windows have been proposed, as reviewed by Harris.\(^{19}\) As an example, the four-term Blackman–Harris window, defined as

\[
w_n = a_0 - a_1 \cos \left( \frac{2\pi n}{N} \right) + a_2 \cos \left( \frac{4\pi n}{N} \right) - a_3 \cos \left( \frac{6\pi n}{N} \right) \quad n = 0, 1, 2, \ldots, N - 1 \quad (83.20)
\]
produces side lobe levels of –92 dB if the coefficients are chosen as $a_0 = 0.35875$, $a_1 = 0.48829$, $a_2 = 0.14128$, $a_3 = 0.01168$\cite{19}. Windows also play an important role in the sampling properties of power spectral estimates, as will be discussed later. Windowing attenuates the contribution of signal samples at the beginning and end of the signal and, therefore, reduces its effective signal duration. This effect is reflected by the equivalent noise bandwidth (ENBW) defined as\cite{19}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure835.png}
\caption{Effect of limited observation time $T$ on the amplitude spectra of a sinusoidal component. (a) Observation of a single harmonic (dashed line) for a limited period of time $T$ is equivalent to the multiplication for the rectangular function represented in (b). The Blackman–Harris window is also represented in (b) (dashed line). (c) Truncating a single harmonic produces spectral estimates smeared by leakage (solid line) as compared with the theoretical result (dashed line) with width equal to the frequency resolution ($f_r \approx 0.004$ Hz).}
\end{figure}
For a rectangular window $ENBW = 1.0$ and for the Blackman–Harris window (Equation 83.20) the corresponding value is $2.0$. The majority of other window shapes have intermediate values of $ENBW$.

**Effects of “Zero-Padding”**

Most FFT algorithms operate with $N = 2^q$ samples, the choice of $q$ is many times critical. Since frequency resolution is inversely proportional to $N$, in many circumstances a value of $q$ leading to $2^q > N$ is preferable to the option of limiting the signal to $N' = 2^{q-1}$ samples with $N' < N$. The most common and simple way of extending a signal to comply with the $2^q$ condition is by zero-padding. For signals with zero mean and with first and last values around zero, this can be accomplished by complementing the signal with $Q$ zeros to achieve the condition $N + Q = 2^q$. For signals with end points different from zero, these values can be used for padding. If initial and final values differ significantly, a linear interpolation from the last to the first point is also a practical option. However, with the application of windowing, most signals will have similar initial and final points and these can be used for zero-padding. As discussed in the next section, zero-padding has important applications for the estimation of correlation functions via FFT. For spectral analysis, it is relatively simple to demonstrate that adding $Q$ zeros corresponds to over sampling the $N$ point original spectrum with a new frequency resolution which is $(N + Q)/N$ times greater than the original resolution. Consequently, although zero-padding does not introduce major distortions, it produces the false illusion of higher resolution than warranted by the available $N$ measured signal samples.

**Phase Spectrum Estimation**

The use of Equation 83.7b to estimate the phase spectrum is fraught with a different kind of problem, resulting from the indetermination of the $\tan^{-1}$ function to discriminate between phase angles with absolute values greater than $\pi$. This problem is illustrated in Figure 83.2b showing that phase angles decrease continuously until reaching $-\pi$ and then “jump” to continue decreasing from the $+\pi$ value. This feature of the phase spectrum is called **wraparound**. Methods to “unwrap” the phase spectrum have been proposed, but a general satisfactory solution to this problem is not available. In some cases the shifting property of the DFT can be used to “rotate” the original signal in time, thus minimizing the slope of the phase spectrum and, consequently, the occurrence of wraparound.

**Sampling Properties of Spectral Estimators**

The most straightforward approach to computing the power spectrum is to use Equation 83.8. This method is known as the **periodogram**. Application is limited to signals which are stationary, meaning stable statistical properties (such as the mean and the variance) along time. For measurements performed on nonstationary systems, such as speech or systems with time-varying parameters, other methods of spectral estimation are available and will be mentioned later. It is possible to demonstrate that when the period of observation $T$ tends to infinity, Equation 83.8 gives an unbiased estimate of the power spectrum. In practice, due to finite values of $T$, the phenomenon of spectral leakage described above will lead to power spectral estimates which are **biased**.

The second inherent problem with the periodogram is the **variance** of the resulting spectral estimates. Assuming $x_n$ to follow a Gaussian distribution, it follows that $a_k$ and $b_k$ will also be Gaussian because Equation 83.2 represents a linear transformation. Since Equations 83.7a and 83.8 involve the sum of two squared Gaussian variates, $P_k$ will follow a $\chi^2$ distribution with two degrees of freedom. In this case the

\[
ENBW = \frac{\sum_{n=0}^{N-1} w_n^2}{\left(\sum_{n=0}^{N-1} w_n\right)^2}
\]
mean and the standard deviation of the power spectral estimate will be the same, independently of the frequency considered. As a consequence, power spectral estimates obtained from Equation 83.8, using a simple sample \(x_n\), should be regarded as highly unreliable. In addition, the variance or standard deviation of this \(\chi^2\) distribution does not decrease with increases in sample duration \(N\). This indicates that the periodogram (Equation 83.8) is an inconsistent estimator of the power spectrum.

For a \(\chi^2\) distribution with \(m\) degrees of freedom, the coefficient of variation is given by

\[
CV[\chi^2_m] = \sqrt{\frac{2m}{m}} = \sqrt{\frac{2}{m}} \tag{83.22}
\]

showing that it is possible to improve the reliability of power spectral estimates by increasing \(m\). This can be achieved by replacing Equation 83.8 by

\[
\hat{P}_k = \frac{1}{L} \sum_{l=1}^{L} c_{kj}^2 \quad k = 0, 1, 2, \ldots N-1 \tag{83.23}
\]

with \(L\) representing a number of separate samples \(x_n\) each with length \(T = N\Delta t\). If only one record of \(x_n\) can be obtained under stationary conditions, it is possible to break down this record into \(L\) segments to obtain an improved estimate of the power spectrum with variance reduced by a factor of \(L\). However, the spectral resolution, given by \(f_r = 1/T\), will be reduced by the same factor \(L\), thus indicating an inescapable compromise between resolution and variance.

A modified periodogram was introduced by Welch\(^2\) consisting of the multiplication of \(x_n\) by a triangular, or other window shape, before computing the individual spectral samples with Equation 83.5. The application of a window justifies overlapping adjacent segments of data by as much as 50%. For a signal with a total duration of \(N\) samples, the combination of overlapping with segmentation (Equation 83.23) can lead to a further reduction of the spectral variance by a factor of \(11/18\).

Averaging \(L\) spectral samples as indicated by Equation 83.23 represents one approach to improve spectral estimation by means of smoothing. A similar effect can be obtained with the correlogram. Equation 83.13 indicates that it is possible to estimate the power spectrum from the autocorrelation function. Limiting the number of shifts of the autocorrelation function to \(p \times N\) is equivalent to smoothing the original spectrum by convolution with the Fourier transform of a Bartlett (triangular) window. As discussed in the next section, the autocorrelation function can also be computed more efficiently with the FFT and it can be shown that in this case it involves a smaller number of numerical operations than the Welch method based on the periodogram.\(^5\)

### FFT Correlation Analysis

Before considering the application of FFT algorithms to compute auto- and cross-correlation functions, it is important to discuss their sampling properties using Equations 83.10 and 83.11 as estimators. Assuming that variables \(x_n\) and \(y_n\) are not defined outside the interval \(0 \leq n \leq N - 1\), it follows from Equation 83.10 that as \(p\) increases and the two functions “slide” past each other, the effective number of summed products is \(N - |p|\) rather than \(N\) as implied by Equations 83.10 and 83.11. For this reason these equations are often rewritten as

\[
r_{xy}(p) = \frac{1}{N - |p|} \sum_{n=0}^{N-|p|+1} x_n y_{n+|p|} \quad p = 0, \pm 1, \pm 2, \ldots \tag{83.24}
\]

The main justification for this modification, however, is that Equations 83.10 and 83.11 lead to biased estimations of correlation functions while Equation 83.24 is unbiased.
Equation 83.24 normally assumes that $x_n$ and $y_n$ are standardized variables with zero mean and unit variance. If the mean values are different from zero, Equation 83.10 and 83.11 will produce distorted estimates with a “pyramid effect” due to the presence of the dc term. However, this effect is compensated for in Equation 83.24 and in this case the effect of the mean value is to add a constant term:

$$r_{xy}(p) = r'_{xy}(p) - m_x m_y$$  \hspace{1cm} (83.25)$$

where $r'_{xy}(p)$ is the cross correlation of variables with mean values $m_x$ and $m_y$, respectively.

Similarly to the DFT, Equations 83.10 and 83.11 involve $N^2$ operations and Equation 83.24 slightly less. Since the autocorrelation and the power spectra constitute a Fourier transform pair (Equation 83.13), the computation of correlation functions can also be sped up by means of an FFT algorithm. For the sake of generality, the cross spectrum of $x_n$ and $y_n$ can be defined as

$$C_{xy}(f_k) = X(f_k) Y^*(f_k)$$  \hspace{1cm} (83.26)$$

with $X(f_k)$ and $Y(f_k)$ representing the Fourier transforms of $x_n$ and $y_n$, respectively. The generalized Wiener–Khintchine theorem then gives the cross-correlation function as

$$r_{xy}(p) = \sum_{k=0}^{N-1} C_{xy}(f_k) e^{-j(2\pi kp/N)}$$  \hspace{1cm} (83.27)$$

Therefore, “fast” correlation functions can be computed using the forward FFT to calculate $X(f_k)$ and $Y(f_k)$ and then the inverse FFT to obtain $r_{xy}(p)$ with Equation 83.27. Obviously, when autocorrelation functions are being computed with this method, only one transform is necessary to obtain the autospectra (Equation 83.13) instead of the cross spectra (Equation 83.26).

When correlation functions are computed with the FFT, it is critical to pay attention again to the periodicity of the transformed variables as an intrinsic property of the DFT. When the two functions in either Equation 83.10 and 83.11 are displaced by $p$ samples, for periodic functions there will be nonzero products outside the range $0 \leq n \leq N - 1$, thus leading to significant errors in the estimated auto- or cross-correlation functions. In this case the resulting estimates are called circular correlations. This error can be avoided by zero-padding the original signals from $n = N$ to $n = 2N - 1$ and computing the FFTs with $2N$ points. The resulting correlation functions will be noncircular and, in the range $0 \leq p \leq N - 1$, will agree with correlations computed with the original Equations 83.10 or 83.11. Finally, to remove bias the results of Equation 83.27 should also be multiplied by $N/(N - |p|)$ to agree with Equation 83.24.

Further Information

Software for FFT special analysis is available from multiple sources. Off-the-shelf software ranges from specialized packages for digital signal processing, such as DADiSP, to statistical packages which include FFT analysis of time series. Mathematical and engineering packages such as MATLAB also include routines for FFT spectral and correlation analysis. For a review of available options see Reference 24. For readers who want to implement their own software, FFT routines can be found in Reference 4, 13 through 15, and 18. Additional references are 25 through 27.

Hardware implementations of FFT algorithms are common in areas requiring real-time spectral analysis as in the case of blood flow velocity measurement with Doppler ultrasound. For a review of hardware implementations see References 11 and 28. Developments in this area follow the pace of change in VLSI technology.

One of the limitations of the FFT is the fact that frequency resolution is the inverse of the signal observation time. Improved resolution can be obtained with parametric methods of spectral analysis and

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their application is particularly relevant when only short segments of data are available or when it is necessary to discriminate between frequency harmonics which are closely spaced in the spectrum. Broadly speaking, parametric methods assume that the data follow spectral densities with a known pole-zero structure of variable complexity, characterized by a given model order. All-zero models correspond to the moving average structure while the all-pole version represents the autoregressive model. The general case is the autoregressive-moving average model (ARMA). For a comprehensive review of these methods see Reference 30; further information and software implementations can be found in References 18 and 27.

Nonstationary signals present a particular problem. In cases where the signal statistical properties change relatively slowly with time, it is possible to select short segments of quasi-stationary data and to use the DFT or parametric methods to estimate the spectra as mentioned previously. However, when these changes in systems parameters or statistical moments are fast in relation to the phenomenon under observation (e.g., speech or seismic data), this approach is not feasible because of the poor frequency resolution resulting from short observation times. Methods proposed to cope with signal nonstationarity often depend on the underlying cause of nonstationary behavior. More general methods, known as time-frequency distributions, are now favored by most investigators. The Wigner–Ville and Choi–Williams transforms are some of the more widely used of these time-frequency distributions. In each case the signal is described by a simultaneous function of time and frequency and hence is graphically represented by a three-dimensional plot having time and frequency as dependent variables.

A different approach to the analysis of nonstationary data is the application of wavelets. This alternative also has advantages in the representation of fast transients and in applications requiring data compression and pattern classification. Similarly to the sine and cosine functions, which are the basis of Fourier analysis, wavelets are orthogonal functions which can be used to decompose and reconstruct signals using a finite set of coefficients obtained by a wavelet transform (WT). The main difference between wavelets and sinusoids, however, is that the former are limited in time. In addition, the complete orthogonal set of wavelets can be obtained simply by expansion (or compression) and scaling of a single function, known as the mother wavelet. Because of their limited time duration wavelets can provide a much more synthetic decomposition of fast transients, or sharp edges in image analysis, than it is possible to obtain with the DFT. Their property of expansion/contraction of a single mother wavelet can also overcome a major limitation of the DFT, that is, to allow good frequency resolution at both low and high frequencies. For applications of the WT and commercially available software see References 34 and 35.

Defining Terms

**Analog-to-digital conversion:** The process of converting a continuous signal to a discrete time sequence of values usually sampled at uniform time intervals.

**Autocorrelation function (ACF):** A measure of longitudinal variability of a signal which can express the statistical dependence of serial samples.

**Correlogram:** Numerical calculation and graphical representation of the ACF or CCF.

**Cross-correlation function (CCF):** A measure of similarity between signals in the time domain which also allows the identification of time delays between transients.

**Cross-spectrum:** The complex product of the power spectra of two different signals.

**Decimation-in-time:** The process of breaking down a time series into subsequences to allow more efficient implementations of the FFT.

**Discrete Fourier transform (DFT):** The usual method to obtain the Fourier series of a discrete time signal.

**Fast Fourier transform (FFT):** Algorithm for the efficient computation of the DFT.

**Periodogram:** A family of methods to estimate the power spectrum using the DFT.

**Power spectrum:** The distribution of signal power as a function of frequency.

**Signal:** Continuous or discrete representation of a variable or measurement as a function of time or other dimension.

**Stationarity:** Property of signals which have statistical moments invariant with time.
Twiddle factors: Exponential term in the DFT whose periodicity allow repeated use and hence considerable savings of computation time in the FFT.

Wraparound: Overflow of phase spectral estimations above $\pi$ due to the uncertainty of the $\tan^{-1}$ function.

Zero-padding: Extension of a signal with zeros, constant values, or other extrapolating functions.

References

Appendix

Units and Conversions

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National Physical Laboratory

This appendix contains several tables that list the SI base units (Table A.1), define the SI base units (Table A.2), list their derived units (Table A.3), list their prefixes (Table A.4), and list their conversion units (Table A.5).

**TABLE A.1** The SI Base Units

<table>
<thead>
<tr>
<th>Base quantity</th>
<th>Name of Base Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>meter</td>
<td>m</td>
</tr>
<tr>
<td>Mass</td>
<td>kilogram</td>
<td>kg</td>
</tr>
<tr>
<td>Time</td>
<td>second</td>
<td>s</td>
</tr>
<tr>
<td>Electric current</td>
<td>ampere</td>
<td>A</td>
</tr>
<tr>
<td>Thermodynamic temperature</td>
<td>kelvin</td>
<td>K</td>
</tr>
<tr>
<td>Amount of substance</td>
<td>mole</td>
<td>mol</td>
</tr>
<tr>
<td>Luminous intensity</td>
<td>candela</td>
<td>cd</td>
</tr>
</tbody>
</table>
TABLE A.2  The International Definitions of the SI Base Units

<table>
<thead>
<tr>
<th>Unit Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>unit of length</strong></td>
<td><strong>(meter)</strong></td>
</tr>
<tr>
<td></td>
<td>The meter is the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second (17th CGPM, 1983, Resolution 1).  Note: The original international prototype, made of platinum-iridium, is kept at the BIPM under conditions specified by the 1st CGPM in 1889.</td>
</tr>
<tr>
<td><strong>unit of mass</strong></td>
<td><strong>(kilogram)</strong></td>
</tr>
<tr>
<td></td>
<td>The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram (3rd CGPM, 1901).</td>
</tr>
<tr>
<td><strong>unit of time</strong></td>
<td><strong>(second)</strong></td>
</tr>
<tr>
<td></td>
<td>The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom (13th CGPM, 1967, Resolution 1).</td>
</tr>
<tr>
<td><strong>unit of electric current</strong></td>
<td><strong>(ampere)</strong></td>
</tr>
<tr>
<td></td>
<td>The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2 \times 10^{-7} newton per meter of length (CIPM, 1946, Resolution 2 approved by the 9th CGPM, 1948). Note: The expression “MKS unit of force” which occurs in the original text has been replaced here by “newton,” a name adopted for this unit by the 9th CGPM (1948), Resolution 7.</td>
</tr>
<tr>
<td><strong>unit of thermodynamic temperature</strong></td>
<td><strong>(kelvin)</strong></td>
</tr>
</tbody>
</table>
|                        | The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water (13th CGPM, 1967, Resolution 4). The 13th CGPM (1967, Resolution 3) also decided that the unit kelvin and its symbol K should be used to express an interval or a difference in temperature. Note: In addition to the thermodynamic temperature (symbol \( T \)), expressed in kelvin, use is also made of Celsius temperature (symbol \( t \)) defined by the equation 

\[
t = T - T_0
\]

where \( T_0 = 273.15 \text{ K} \) by definition. To express Celsius temperature, the unit “degree Celsius” which is equal to the unit “kelvin” is used; in this case “degree Celsius” is a special name used in place of “kelvin.” An interval or difference of Celsius temperature can, however, be expressed in kelvins as well as degrees Celsius. |
| **unit of amount of substance** | **(mole)**                                                                                                                                                                                               |
|                        | 1. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12.  
2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles. In the definition of the mole, it is understood that unbound atoms of carbon-12, at rest, and in their ground state, are referred to. Note: This definition specifies at the same time the nature of the quantity whose unit is the mole. |
| **Unit of luminous intensity** | **(candela)**                                                                                                                                                                                             |
|                        | The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency \( 540 \times 10^{12} \text{ hertz} \) and that has a radiant intensity in that direction of \( (1/683) \text{ watt per steradian} \) (16th CGPM, 1979, resolution 3). |

\( a \) The U.S. denotes the unit of length by “meter” in place of the international usage of “meter”.  
\( b \) CGPM: Conférence Général de Poids et Mesures; CIPM: Comité International des Poids et Mesures.  
\( c \) BIPM: Bureau International des Poids et Mesures.
### TABLE A.3 SI Derived Units with Special Names*

<table>
<thead>
<tr>
<th>Derived quantity</th>
<th>Name</th>
<th>Symbol</th>
<th>Expressed in Terms of Other Units</th>
<th>Expressed in Terms of SI Base Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane angle</td>
<td>radian</td>
<td>rad</td>
<td>m⁻¹</td>
<td>m⁻²</td>
</tr>
<tr>
<td>Solid angle</td>
<td>steradian</td>
<td>sr</td>
<td>m²</td>
<td>m²⁻¹</td>
</tr>
<tr>
<td>Frequency</td>
<td>hertz</td>
<td>Hz</td>
<td>s⁻¹</td>
<td></td>
</tr>
<tr>
<td>Force</td>
<td>newton</td>
<td>N</td>
<td>m kg s⁻²</td>
<td></td>
</tr>
<tr>
<td>Pressure, stress</td>
<td>pascal</td>
<td>Pa</td>
<td>N m⁻²</td>
<td>m⁻¹ kg s⁻²</td>
</tr>
<tr>
<td>Energy, work, quantity of heat</td>
<td>joule</td>
<td>J</td>
<td>m² kg s⁻²</td>
<td></td>
</tr>
<tr>
<td>Power, radiant flux</td>
<td>watt</td>
<td>W</td>
<td>m² kg s⁻³</td>
<td></td>
</tr>
<tr>
<td>Electric charge, quantity of electricity</td>
<td>coulomb</td>
<td>C</td>
<td></td>
<td>s A</td>
</tr>
<tr>
<td>Electric potential, potential difference, electromotive force</td>
<td>volt</td>
<td>V</td>
<td>W/A</td>
<td>m² kg s⁻³ A⁻¹</td>
</tr>
<tr>
<td>Capacitance</td>
<td>farad</td>
<td>F</td>
<td>C/V</td>
<td>m⁻² kg⁻¹ s⁴ A²</td>
</tr>
<tr>
<td>Electric resistance</td>
<td>ohm</td>
<td>Ω</td>
<td>V/A</td>
<td>m² kg s⁻³ A⁻²</td>
</tr>
<tr>
<td>Electric conductance</td>
<td>siemens</td>
<td>S</td>
<td>A.V</td>
<td>m² kg s⁻³ A⁻²</td>
</tr>
<tr>
<td>Magnetic flux</td>
<td>weber</td>
<td>Wb</td>
<td>V s</td>
<td>m² kg s⁻² A⁻¹</td>
</tr>
<tr>
<td>Magnetic flux density</td>
<td>tesla</td>
<td>T</td>
<td>Wb/m²</td>
<td>kg s⁻² A⁻¹</td>
</tr>
<tr>
<td>Inductance</td>
<td>henry</td>
<td>H</td>
<td>Wb/A</td>
<td>m² kg s⁻³ A⁻²</td>
</tr>
<tr>
<td>Celsius temperature</td>
<td>degree Celsius</td>
<td>°C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Luminous flux</td>
<td>lumen</td>
<td>lm</td>
<td>cd sr</td>
<td>cd m⁻² m⁻² = cd</td>
</tr>
<tr>
<td>Illuminance</td>
<td>lux</td>
<td>lx</td>
<td>m⁻² cd sr</td>
<td>m⁻² cd</td>
</tr>
<tr>
<td>Activity (referred to a radio nuclide)</td>
<td>becquerel</td>
<td>Bq</td>
<td></td>
<td>s⁻¹</td>
</tr>
<tr>
<td>Absorbed dose, specific energy</td>
<td>gray</td>
<td>G</td>
<td>J/kg</td>
<td>m⁻² s⁻²</td>
</tr>
<tr>
<td>imparted, kerma</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dose equivalent, ambient dose</td>
<td>sievert</td>
<td>Sr</td>
<td>J/kg</td>
<td>m⁻² s⁻²</td>
</tr>
</tbody>
</table>

* Note that when a unit is named after a person the symbol takes a capital letter and the name takes a lowercase letter.

### TABLE A.4 SI Prefixes*

<table>
<thead>
<tr>
<th>Factor</th>
<th>Prefix</th>
<th>Symbol</th>
<th>Factor</th>
<th>Prefix</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>10²⁴</td>
<td>yotta</td>
<td>Y</td>
<td>10⁻¹</td>
<td>deci</td>
<td>d</td>
</tr>
<tr>
<td>10²¹</td>
<td>zetta</td>
<td>Z</td>
<td>10⁻²</td>
<td>centi</td>
<td>c</td>
</tr>
<tr>
<td>10¹⁸</td>
<td>exa</td>
<td>E</td>
<td>10⁻³</td>
<td>milli</td>
<td>m</td>
</tr>
<tr>
<td>10¹⁵</td>
<td>peta</td>
<td>P</td>
<td>10⁻⁶</td>
<td>micro</td>
<td>µ</td>
</tr>
<tr>
<td>10¹²</td>
<td>tera</td>
<td>T</td>
<td>10⁻⁹</td>
<td>nano</td>
<td>n</td>
</tr>
<tr>
<td>10⁹</td>
<td>giga</td>
<td>G</td>
<td>10⁻¹²</td>
<td>pico</td>
<td>p</td>
</tr>
<tr>
<td>10⁶</td>
<td>mega</td>
<td>M</td>
<td>10⁻¹⁵</td>
<td>femto</td>
<td>f</td>
</tr>
<tr>
<td>10³</td>
<td>kilo</td>
<td>k</td>
<td>10⁻¹⁸</td>
<td>atto</td>
<td>a</td>
</tr>
<tr>
<td>10²</td>
<td>hecto</td>
<td>h</td>
<td>10⁻²¹</td>
<td>zepto</td>
<td>z</td>
</tr>
<tr>
<td>10</td>
<td>deca</td>
<td>da</td>
<td>10⁻²⁴</td>
<td>yocto</td>
<td>y</td>
</tr>
</tbody>
</table>

* The 11th CGPM (1960, Resolution 12) adopted a first series of prefixes and symbols of prefixes to form the names and symbols of the decimal multiples and submultiples of SI units. Prefixes for 10⁻¹⁵ and 10⁻¹⁸ were added by the 12th CGPM (1964, Resolution 8), those for 10⁻²⁵ and 10⁻²⁸ by the 15th CGPM (1975, Resolution 10), and those for 10⁷, 10⁴, 10³, and 10² were proposed by the CIPM (1990) for approval by the 19th CGPM (1991).
**TABLE A.5** Conversion Factors from English Measures to SI Units

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. Acceleration</strong></td>
<td></td>
</tr>
<tr>
<td>Acceleration of free fall, standard gravity</td>
<td>9.806 65 m/s²</td>
</tr>
<tr>
<td>1 ft/s²</td>
<td>0.304 8 m/s²</td>
</tr>
<tr>
<td>1 gal</td>
<td>0.01 m/s²</td>
</tr>
<tr>
<td><strong>2. Angle</strong></td>
<td></td>
</tr>
<tr>
<td>1 second (&quot;)</td>
<td>4.848 1 × 10⁻⁶ rad</td>
</tr>
<tr>
<td>1 minute (')</td>
<td>2.908 9 × 10⁻⁴ rad</td>
</tr>
<tr>
<td>1 degree (°)</td>
<td>0.0174 532 rad</td>
</tr>
<tr>
<td>1 rad</td>
<td>206 264.8°</td>
</tr>
<tr>
<td><strong>3. Area</strong></td>
<td></td>
</tr>
<tr>
<td>1 barn (b)</td>
<td>10⁻²⁸ m²</td>
</tr>
<tr>
<td>1 in.²</td>
<td>6.451 6 × 10⁻⁴ m²</td>
</tr>
<tr>
<td>1 ft²</td>
<td>0.092 903 04 m²</td>
</tr>
<tr>
<td>1 yd²</td>
<td>0.836 127 36 m²</td>
</tr>
<tr>
<td>1 are</td>
<td>100 m²</td>
</tr>
<tr>
<td>1 acre [43560 (statute ft)²]</td>
<td>4046.86 m²</td>
</tr>
<tr>
<td>1 hectare</td>
<td>10 000 m²</td>
</tr>
<tr>
<td>1 mi²</td>
<td>2.590 0 × 10⁶ m²</td>
</tr>
<tr>
<td>1 square mile (based on U.S. survey foot)</td>
<td>2.589 998 km²</td>
</tr>
<tr>
<td><strong>4. Concentration, Density, Mass Density</strong></td>
<td></td>
</tr>
<tr>
<td>1 grain/gal (U.S.)</td>
<td>0.017 118 kg/m³</td>
</tr>
<tr>
<td>1 lb/ft³</td>
<td>16.018 46 kg/m³</td>
</tr>
<tr>
<td>1 lb/gal (U.S.)</td>
<td>119.826 4 kg/m³</td>
</tr>
<tr>
<td>1 short ton/yd³</td>
<td>1186.6 kg/m³</td>
</tr>
<tr>
<td>1 long ton/yd³</td>
<td>1328.9 kg/m³</td>
</tr>
<tr>
<td>1 oz (avdp)/in.³</td>
<td>1730.0 kg/m³</td>
</tr>
<tr>
<td>1 oz (avd)/gal (U.S.)</td>
<td>7489.152 kg/m³</td>
</tr>
<tr>
<td>1 lb/in.³</td>
<td>27 680 kg/m³</td>
</tr>
<tr>
<td><strong>5. Energy</strong></td>
<td></td>
</tr>
<tr>
<td>1 ft lbf</td>
<td>1.355 818 J</td>
</tr>
<tr>
<td>1 cal₀ (thermochemical calorie)</td>
<td>4.184 J</td>
</tr>
<tr>
<td>1 cal₁₅ (15°C calorie)</td>
<td>4.185 5 J</td>
</tr>
<tr>
<td>1 cal₂₀</td>
<td>4.186 8 J</td>
</tr>
<tr>
<td>1 kilocalorie (nutrition)c</td>
<td>4.186.8 J</td>
</tr>
<tr>
<td>1 watt second (W s)</td>
<td>1 J</td>
</tr>
<tr>
<td>1 watt hour (W h)</td>
<td>3600 J</td>
</tr>
<tr>
<td>1 therm (EC)</td>
<td>1.055 06 × 10⁶ J</td>
</tr>
<tr>
<td>1 therm (U.S.)</td>
<td>1.054 804 × 10⁶ J</td>
</tr>
<tr>
<td>1 ton TNT (equivalent)</td>
<td>4.184 × 10⁹ J</td>
</tr>
<tr>
<td>1 BT₀₀</td>
<td>1 054.350 J</td>
</tr>
<tr>
<td>1 Btu₀₀</td>
<td>1 054.728 J</td>
</tr>
<tr>
<td>1 Btu₀₁</td>
<td>1 055.055 852 62 J</td>
</tr>
<tr>
<td>1 quad (= 10¹⁵ Btu)</td>
<td>≈10¹⁸ J = 1 EJ</td>
</tr>
<tr>
<td><strong>6. Force</strong></td>
<td></td>
</tr>
<tr>
<td>1 dyne</td>
<td>10⁻⁵ N</td>
</tr>
<tr>
<td>1 ounce-force</td>
<td>0.278 013 9 N</td>
</tr>
<tr>
<td>1 pound-force</td>
<td>4.448 222 N</td>
</tr>
<tr>
<td>1 kilogram-force</td>
<td>9.806 65 N</td>
</tr>
<tr>
<td>1 kip (1000 lbf)</td>
<td>4448.222 N</td>
</tr>
<tr>
<td>1 ton-force (2000 lbf)</td>
<td>8.896 443 N</td>
</tr>
</tbody>
</table>
### 7. Fuel consumption

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 gallon (U.S.) per horsepower hour</td>
<td>$1.410 \times 10^{-6} \text{ m}^{3}/\text{J}$</td>
</tr>
<tr>
<td>1 gallon (U.S.)/mile</td>
<td>2.352 15 l/km</td>
</tr>
<tr>
<td>1 gallon (U.K.)/mile</td>
<td>2.824 81 l/km</td>
</tr>
<tr>
<td>1 mile/gallon (U.S.), mpg</td>
<td>0.425 144 km/l</td>
</tr>
<tr>
<td>1 mile/gallon (U.K.)</td>
<td>0.354 006 km/l</td>
</tr>
<tr>
<td>1 pound per horsepower</td>
<td>1.689 659 $\times 10^{-7}$ kg/J</td>
</tr>
<tr>
<td>1 l/(100 km)</td>
<td>235.215/(mpg) (U.S.)</td>
</tr>
</tbody>
</table>

### 8. Length

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 fermi</td>
<td>$10^{-15}$ m = 1 fm</td>
</tr>
<tr>
<td>1 angstrom (Å)</td>
<td>$10^{-10}$ m</td>
</tr>
<tr>
<td>1 microinch</td>
<td>$2.54 \times 10^{-6}$ m</td>
</tr>
<tr>
<td>1 mil</td>
<td>$2.54 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>1 point (pt) [0.013837 in]^2</td>
<td></td>
</tr>
<tr>
<td>1 pica (12 pt)</td>
<td>4.217 5 mm</td>
</tr>
<tr>
<td>1 inch (in.)</td>
<td>0.025 4 m</td>
</tr>
<tr>
<td>1 hand (4 in.)</td>
<td>0.101 6 m</td>
</tr>
<tr>
<td>1 foot (12 in.) (0.999998 statute ft.)</td>
<td>0.304 8 m</td>
</tr>
<tr>
<td>1 foot (U.S. survey)</td>
<td>0.304 800 6 m</td>
</tr>
<tr>
<td>1 statute foot [(1200/3937) m]</td>
<td>0.304 800 6 m</td>
</tr>
<tr>
<td>1 yard (yd)</td>
<td>0.914 4 m</td>
</tr>
<tr>
<td>1 fathom (6 ft, U.S. survey)</td>
<td>1.828 8 m</td>
</tr>
<tr>
<td>1 rod (16.5 statute ft)</td>
<td>5.029 2 m</td>
</tr>
<tr>
<td>1 chain (4 rod)</td>
<td>20.116 8 m</td>
</tr>
<tr>
<td>1 furlong (80 chain)</td>
<td>201.168 m</td>
</tr>
<tr>
<td>1 mile (8 furlong, 5280 ft)</td>
<td>1609.344 m</td>
</tr>
<tr>
<td>1 statute mile (8 furlong, 5280 statute ft)</td>
<td>1609.347 2 m</td>
</tr>
<tr>
<td>1 nautical mile (international)^e</td>
<td>1852 m</td>
</tr>
<tr>
<td>1 light year^t</td>
<td>$9.640 73 \times 10^{15}$ m</td>
</tr>
</tbody>
</table>

### 9. Light

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 foot-candle</td>
<td>10.763 91 lx</td>
</tr>
<tr>
<td>1 phot</td>
<td>10 000 lx</td>
</tr>
<tr>
<td>1 cd/in.²</td>
<td>1550.003 cd/m²</td>
</tr>
<tr>
<td>1 foot-lambert</td>
<td>3.426 259 cd/m²</td>
</tr>
<tr>
<td>1 lambert</td>
<td>3183.099 cd/m²</td>
</tr>
<tr>
<td>1 stilb</td>
<td>10 000 cd/m²</td>
</tr>
</tbody>
</table>

### 10. Mass

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 pound (avdp.) (lb) (7000 gr)</td>
<td>0.453 592 37 kg</td>
</tr>
<tr>
<td>1 pound (troy) (5760 gr)</td>
<td>0.373 241 721 6 kg</td>
</tr>
<tr>
<td>1 grain (gr)</td>
<td>64.798 91 mg</td>
</tr>
<tr>
<td>1 scruple (20 gr)</td>
<td>1.296 0 g</td>
</tr>
<tr>
<td>1 pennyweight (24 gr)</td>
<td>1.555 174 g</td>
</tr>
<tr>
<td>1 dram (60 gr)</td>
<td>3.887 9 g</td>
</tr>
<tr>
<td>1 ounce (avdp) (437.5 gr)</td>
<td>28.349 52 g</td>
</tr>
<tr>
<td>1 ounce (troy) (480 gr)</td>
<td>31.103 48 g</td>
</tr>
<tr>
<td>1 carat (metric)</td>
<td>0.2 g</td>
</tr>
<tr>
<td>1 stone (14 lb)</td>
<td>6.350 29 kg</td>
</tr>
<tr>
<td>1 slug</td>
<td>14.593 9 kg</td>
</tr>
<tr>
<td>1 hundredweight (long)</td>
<td>50.802 35 kg</td>
</tr>
<tr>
<td>1 ton (short) (2000 lb)</td>
<td>907.184 7 kg</td>
</tr>
<tr>
<td>1 ton (long) (2240 lb)</td>
<td>1016.047 kg</td>
</tr>
<tr>
<td></td>
<td>1.016 047 t</td>
</tr>
</tbody>
</table>
### TABLE A.5 Conversion Factors from English Measures to SI Units  
(continued)

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mass per Unit Length</strong></td>
<td></td>
</tr>
<tr>
<td>1 tex</td>
<td>$10^{-6}$ kg/m</td>
</tr>
<tr>
<td>1 denier</td>
<td>$1.111 \times 10^{-7}$ kg/m</td>
</tr>
<tr>
<td>1 pound per foot</td>
<td>1.488 164 kg/m</td>
</tr>
<tr>
<td>1 pound per inch</td>
<td>17.857 97 kg/m</td>
</tr>
<tr>
<td>1 ton/mile</td>
<td>0.631 342 Mg/km</td>
</tr>
<tr>
<td>1 ton/1000 yd</td>
<td>1.111 6 kg/m</td>
</tr>
<tr>
<td>1 lb/ft</td>
<td>1.488 16 kg/m</td>
</tr>
<tr>
<td><strong>Mass per Unit Area</strong></td>
<td></td>
</tr>
<tr>
<td>1 ton/mile²</td>
<td>3.922 98 kg/ha</td>
</tr>
<tr>
<td>1 ton/acre</td>
<td>2510.71 kg/ha</td>
</tr>
<tr>
<td>1 oz/yd²</td>
<td>33.905 7 g/m²</td>
</tr>
<tr>
<td><strong>Mass Carried × Distance (traffic factor)</strong></td>
<td></td>
</tr>
<tr>
<td>1 ton mile</td>
<td>1635.17 kg km</td>
</tr>
<tr>
<td><strong>Mass carried × Distance/Volume (traffic factor)</strong></td>
<td></td>
</tr>
<tr>
<td>1 ton mile/gal (U.S.)</td>
<td>431.967 $6 \text{ Mg km/m}^3$</td>
</tr>
<tr>
<td><strong>11. Power</strong></td>
<td></td>
</tr>
<tr>
<td>1 erg/s</td>
<td>$10^{-7}$ W</td>
</tr>
<tr>
<td>1 ft lbf/h</td>
<td>3.766 161 $\times 10^{-4}$ W</td>
</tr>
<tr>
<td>(1 Btu/ (h ft))</td>
<td>1.000 669 Btu/hr</td>
</tr>
<tr>
<td>1 metric horsepower (force de cheval)</td>
<td>735.498 $8 \text{ W}$</td>
</tr>
<tr>
<td>1 horsepower (550 ft lbf/s)</td>
<td>745.70 W</td>
</tr>
<tr>
<td>1 electric horsepower</td>
<td>746 $W$</td>
</tr>
<tr>
<td><strong>12. Pressure, Stress</strong></td>
<td></td>
</tr>
<tr>
<td>1 standard atmosphere</td>
<td>101 325 Pa</td>
</tr>
<tr>
<td>1 dyne/cm²</td>
<td>0.1 Pa</td>
</tr>
<tr>
<td>1 torr [(101th 325/760) Pa]</td>
<td>133.322 $4 \text{ Pa}$</td>
</tr>
<tr>
<td>1 N/cm²</td>
<td>10 000 Pa</td>
</tr>
<tr>
<td>1 bar</td>
<td>100 000 Pa</td>
</tr>
<tr>
<td>1 lbf/ft²</td>
<td>47.880 26 Pa</td>
</tr>
<tr>
<td>1 lbf/in² (psi)</td>
<td>6894.8 Pa</td>
</tr>
<tr>
<td>1 kgf/cm²</td>
<td>98 066.5 Pa</td>
</tr>
<tr>
<td>1 cm water (4°C)</td>
<td>98.063 7 Pa</td>
</tr>
<tr>
<td>1 mm of mercury (0°C)</td>
<td>133.322 $4 \text{ Pa}$</td>
</tr>
<tr>
<td>1 in of water (39.2°F)</td>
<td>249.082 Pa</td>
</tr>
<tr>
<td>1 in of mercury (60°F)</td>
<td>3376.85 Pa</td>
</tr>
<tr>
<td>1 ft water (39.2°F)</td>
<td>2988.98 Pa</td>
</tr>
<tr>
<td><strong>13. Thermal Quantities</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Fixed Points</strong></td>
<td></td>
</tr>
<tr>
<td>Triple point of natural water: $T_w$</td>
<td>273.16 K</td>
</tr>
<tr>
<td>Zero Celsius (= $T_0 = t_o$)</td>
<td>273.15 K $= 32^\circ \text{F}$</td>
</tr>
<tr>
<td><strong>Temperature Conversions</strong></td>
<td></td>
</tr>
<tr>
<td>Kelvin to Rankine ($T_k$):</td>
<td>$T = (5/9)T_k$</td>
</tr>
<tr>
<td>Kelvin to Celsius</td>
<td>$t = T - T_0$</td>
</tr>
<tr>
<td>Kelvin to Fahrenheit</td>
<td>$t_f = (9/5)(T - T_0) + t_{10}$</td>
</tr>
<tr>
<td>Celsius to Fahrenheit</td>
<td>$t_f = (9/5) t + t_{10}$</td>
</tr>
<tr>
<td>[Numerically: 5($t_1$ + 40) = 9($t_f$ + 40), where $t_1$ and $t_f$ are the numerical values of the Celsius and Fahrenheit temperatures respectively.]</td>
<td></td>
</tr>
<tr>
<td>Unit</td>
<td>Equivalent</td>
</tr>
<tr>
<td>---------------------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td><strong>Temperature Interval Conversions</strong></td>
<td></td>
</tr>
<tr>
<td>1 degree centigrade</td>
<td>1 degree Celsius, denoted 1°C</td>
</tr>
<tr>
<td>1°C</td>
<td>1 K</td>
</tr>
<tr>
<td>1°F</td>
<td>(1/1.8) K</td>
</tr>
<tr>
<td>1°F</td>
<td>(1/1.8) K</td>
</tr>
</tbody>
</table>

**Other Thermal Quantities**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Btu/h (h)</td>
<td>0.292 875 W</td>
</tr>
<tr>
<td>1 Btu/h (h)</td>
<td>0.293 071th 1 W</td>
</tr>
<tr>
<td>1 cal/s</td>
<td>4.186 8 W</td>
</tr>
<tr>
<td>1 cal/s</td>
<td>4.184 W</td>
</tr>
<tr>
<td>1 cal/(g °C)</td>
<td>4186.8 J/(kg K)</td>
</tr>
<tr>
<td>1 Btu ft/(ft² h °F)</td>
<td>1.730 735 W m⁻¹K⁻¹</td>
</tr>
<tr>
<td>1 Btu in/(ft² s °F)</td>
<td>519.220 4 W m⁻¹K⁻¹</td>
</tr>
<tr>
<td>1 clo</td>
<td>0.155 m² K/kW</td>
</tr>
<tr>
<td>1°F h ft/Btu</td>
<td>0.176 110 2 K m²/W</td>
</tr>
<tr>
<td>1°F h ft/Btu-in</td>
<td>6.933 472 K m/W</td>
</tr>
<tr>
<td>1 Btu/lb °F = 1 cal/(g °C)</td>
<td>4186.8 J/kg.K</td>
</tr>
</tbody>
</table>

**14. Torque, Moment of Force**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 dyne·cm</td>
<td>10⁻⁷ N m</td>
</tr>
<tr>
<td>1 kgf·m</td>
<td>9.806 65 N m</td>
</tr>
<tr>
<td>1 ozf·in</td>
<td>0.007 061 552 N m</td>
</tr>
<tr>
<td>1 lbf·in</td>
<td>0.112 904 8 N m</td>
</tr>
<tr>
<td>1 lbf·ft</td>
<td>1.355 818 N m</td>
</tr>
</tbody>
</table>

**15. Velocity (includes speed)**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 foot per hour</td>
<td>8.466 667 × 10⁻³ m/s</td>
</tr>
<tr>
<td>1 foot per minute</td>
<td>0.005 08 m/s</td>
</tr>
<tr>
<td>1 knot (nautical mile per hour)</td>
<td>0.514 444 m/s</td>
</tr>
<tr>
<td>1 mile per hour (mi/h)</td>
<td>0.447 04 m/s</td>
</tr>
</tbody>
</table>

**16. Viscosity**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 poise</td>
<td>0.1 Pa s</td>
</tr>
<tr>
<td>1 ft²/s</td>
<td>0.092 903 04 m²/s</td>
</tr>
<tr>
<td>1 lbf/(ft s)</td>
<td>1.488 164 Pa s</td>
</tr>
<tr>
<td>1 lbf/(ft h)</td>
<td>4.133 789 × 10⁻⁴ Pa s</td>
</tr>
<tr>
<td>1 lbf/s/ft²</td>
<td>47.880 26 Pa s</td>
</tr>
<tr>
<td>1 lbf/s/in²</td>
<td>6894.757 Pa s</td>
</tr>
<tr>
<td>1 rhe</td>
<td>10 Pa s⁻¹</td>
</tr>
<tr>
<td>1 slug/ft s</td>
<td>47.880 26 Pa s</td>
</tr>
<tr>
<td>1 stokes, St</td>
<td>1.0 × 10⁻⁴ m³</td>
</tr>
</tbody>
</table>

**17. Volume (includes capacity)**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 stere, st</td>
<td>1 m³</td>
</tr>
<tr>
<td>1 liter</td>
<td>0.001 m³</td>
</tr>
<tr>
<td>1 ft³</td>
<td>0.028 316 8 m³</td>
</tr>
<tr>
<td>1 in.³</td>
<td>1.638 7 × 10⁻⁴ m³</td>
</tr>
<tr>
<td>1 board foot</td>
<td>2.359 7 × 10⁻³ m³</td>
</tr>
<tr>
<td>1 acre-foot</td>
<td>1233.48 m²</td>
</tr>
<tr>
<td>1 dram (U.S. fluid)</td>
<td>3.696 7 × 10⁻⁶ m³</td>
</tr>
<tr>
<td>1 gill (U.S.)</td>
<td>1.182 941 × 10⁻⁴ m³</td>
</tr>
<tr>
<td>1 ounce (U.S. fluid)</td>
<td>2.957 353 × 10⁻⁴ m³</td>
</tr>
<tr>
<td>1 teaspoon (tsp)</td>
<td>4.928 822 × 10⁻⁵ m³</td>
</tr>
<tr>
<td>1 tablespoon (tbsp)</td>
<td>1.478 767 × 10⁻⁵ m³</td>
</tr>
<tr>
<td>1 pint (U.S. fluid)</td>
<td>4.731 765 × 10⁻⁴ m³</td>
</tr>
</tbody>
</table>
### TABLE A.5 Conversion Factors from English Measures to SI Units (continued)

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 quart (U.S. fluid)</td>
<td>$9.463 \times 10^{-4}$ m$^3$</td>
</tr>
<tr>
<td>1 gallon (U.S. liquid) [231 in.$^3$]</td>
<td>$3.785 \times 10^{-3}$ m$^3$</td>
</tr>
<tr>
<td>1 wine barrel (bbl) [31.5 gal (U.S.)]</td>
<td>0.119 240 m$^3$</td>
</tr>
<tr>
<td>1 barrel (petroleum, 42 gal, U.S.), bbl</td>
<td>0.158 987</td>
</tr>
<tr>
<td>1 ounce (U.K. fluid)</td>
<td>$2.841 \times 10^{-4}$ m$^3$</td>
</tr>
<tr>
<td>1 gill (Canada &amp; U.K.)</td>
<td>$1.420 \times 10^{-3}$ m$^3$</td>
</tr>
<tr>
<td>1 gallon (Canada &amp; U.K.)</td>
<td>$4.546 \times 10^{-3}$ m$^3$</td>
</tr>
<tr>
<td>1 pint (U.S. dry)</td>
<td>$1.000 \times 10^{-3}$ m$^3$</td>
</tr>
<tr>
<td>1 quart (U.S. dry)</td>
<td>$1.101 \times 10^{-3}$ m$^3$</td>
</tr>
<tr>
<td>1 gallon (U.S. dry)</td>
<td>$4.404 \times 10^{-3}$ m$^3$</td>
</tr>
<tr>
<td>1 peck</td>
<td>8.809 768 \times 10^{-3}$ m$^3$</td>
</tr>
<tr>
<td>1 bushel (U.S.) [2150.42 in.$^3$]</td>
<td>$3.523 \times 10^{-2}$ m$^3$</td>
</tr>
</tbody>
</table>

The conversion factor for a compound unit is usually not given here if it may easily be derived from simpler conversions; e.g., the conversion factors for "$ft/s$" to "$m/s$" or "$ft/s^2$" to "$m/s^2$" are not given, since they may be obtained from the conversion factor for "$ft$." Values are given to five or six significant digits except for exact values, which are usually indicated in bold type. A few former cgs measures are also included.

a The International Steam Table calorie of 1956.

b In practice the prefix kilo is usually omitted. The kilogram calorie or large calorie is an obsolete term for the kilocalorie which is used to express the energy content of foods.

c Typographer’s definition, 1886.

d Originally, in 1929, the International nautical mile.

e Based on 1 day = 86,400 s and 1 Julian century = 36,525 days.

f Post 1964 value, SI symbol l or L. Between 1901 and 1964 the liter was defined as 1.0000028 dm$^3$.

g Although often given, it is doubtful whether normal usage justifies this accuracy. In Europe and elsewhere the teaspoon and tablespoon are usually exactly 5 mL and 15 mL, respectively.

### References

1. CIPM, Procès-Verbaux CIPM, 49th Session, 1960, pp 71–72; Comptes Rendues, 11th CGPM, 1960, p. 85


